

FINITE ELEMENT METHOD – BASIC DEFINITIONS

1. DISCRETIZATION PROCESS:

The various considerations to be taken in the discretization process are as given below:

1.1 Type of Elements or Various Elements Shapes or Element Library: The discretization of the structure or body into finite elements forms the first step in finite element analysis of a complicated structural system. These elements coincide with the geometry of the structure and represent the geometry and mechanical properties in the regions. Therefore, the first decision the engineer must make is to select the shape of the basic elements to be used in the analysis. The selection of finite element in FEM is depends on the following factors. They are:

1. Dimension of the problem.
2. Shape of the body or continuum.
3. Accuracy of the results required.
4. Cost and time for the analysis.

Fig. 1.1 shows some typical elements used to discretize the continuum.

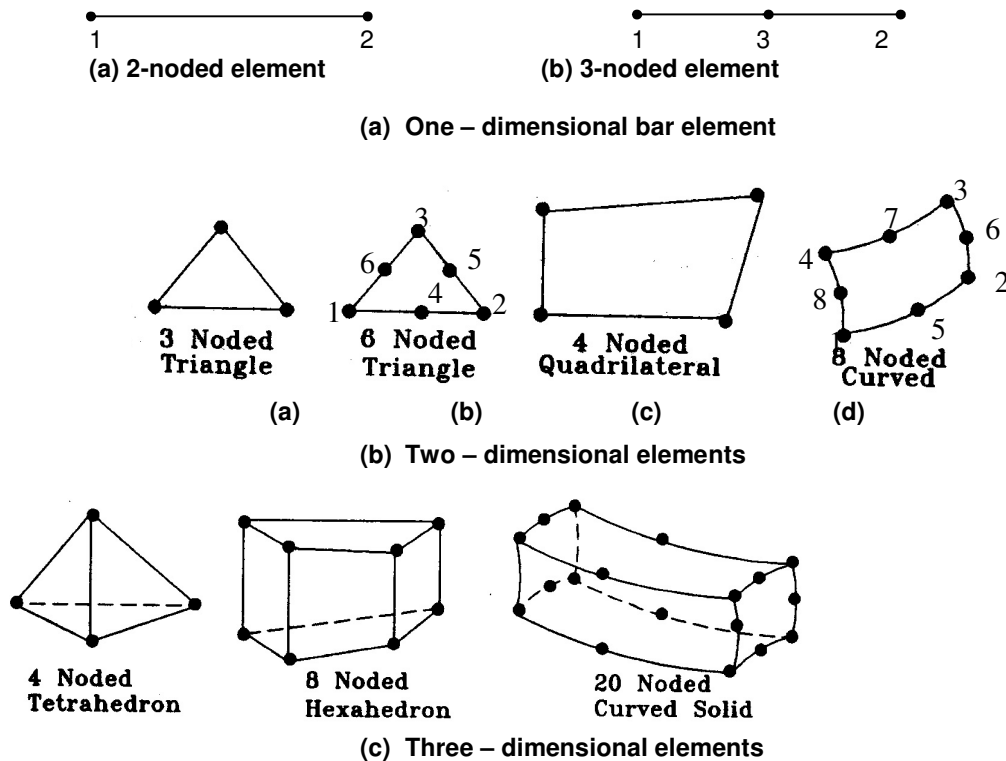


Fig. 1.1: Typical elements

1D Elements: When the geometry, material properties and field variable of the problem is described in terms of only one spatial coordinate (for example x- coordinate), then line elements shown in Fig. 1(a) can be used. A one – dimensional element may be represented by a straight line whose ends are nodal points as shown in Fig. 1(a). These nodal points, numbered 1 and 2 are called *external nodes* because they represent connecting points to the adjacent elements. Some applications require additional nodal point such as node-3 as shown in Fig. 1a(b). Because no connection to other elements occurs at this node, it is called *internal node*.

2D Elements: When the configuration and other parameters of the problems are described in terms of two spatial coordinates, then two-dimensional elements are used. Triangular or quadrilateral elements (Fig.1b) are used for two – dimensional problems such as plane stress or plain strain or plate bending problems. The corner nodes of these elements are called *primary external nodes* and additional nodes occur on the sides of the elements, like 4, 5 and 6 in 6 – noded triangular element or 5, 6, 7 and 8 in 8 – noded curved element are called *secondary external nodes*. Some times *internal node* such as 7 as shown in Fig.1b (b) is also used for these elements.

3D Elements: When the geometry, material properties and other parameters of the problems are described by three spatial coordinates, then three-dimensional elements shown in Fig. 1(c) are used. For 3 – dimensional problems, tetrahedron or hexahedron elements are used. A tetrahedron has 4 primary external nodes and hexahedron has 8 primary external nodes.

1.2 Size of Elements: The size of elements influences the convergence of the solution directly and hence it has to be chosen with care. If the size of the elements is small, the final solution is expected to be more accurate. However, the use of elements of smaller size will take more computational time. Sometimes, we may have to use elements of different sizes in the same body. For example, in the case of stress analysis of the plate in Figure 1.2(a), the size of all the elements can be approximately the same, as shown in Figure 1.2(b). However, in the case of stress analysis of a plate with a hole shown in Figure 1.2(c), elements of different sizes have to be used, as shown in Figure 1.2(d). The size of elements has to be very small near the hole (where stress concentration is expected) compared to far away places. In general, whenever steep gradients of the field variable are expected, we have to use a finer mesh in those regions. Another characteristic related to the size of elements that affects the finite element solution is the **aspect ratio** of the elements. The aspect ratio describes the shape of the element in the assemblage of elements. For two-dimensional elements, the aspect ratio is taken as the ratio of the largest dimension of the element to the smallest dimension. Elements with an aspect ratio of nearly unity generally yield best results.

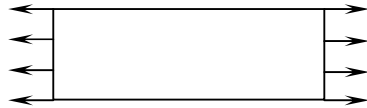


Fig.1.2(a) Original Structure

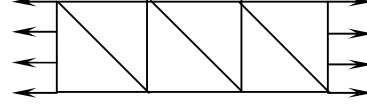


Fig.1.2(b) Finite Element Discretization

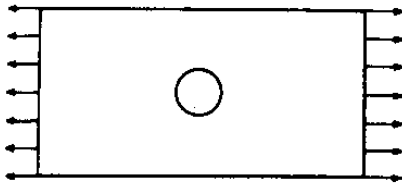


Fig. 1.2(c) Original Structure

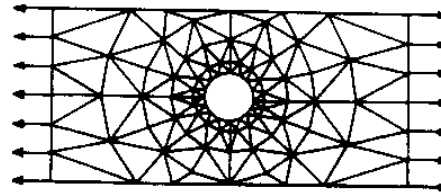
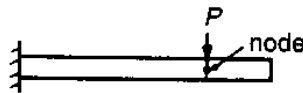
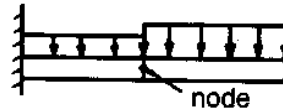


Fig. 1.2(d) Finite Element Discretization

1.3 Location of Nodes: If the body has no abrupt changes in geometry, material properties, and external conditions (e.g., load and temperature), the body can be divided into equal subdivisions and hence the spacing of the nodes can be uniform. On the other hand, if there are any discontinuities in the problem, nodes have to be introduced at these discontinuities, as shown in Fig 1.3



(i) Concentrated load on a beam

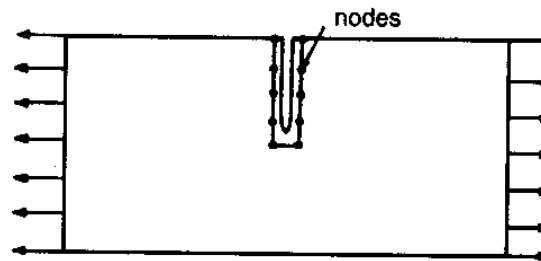


(ii) Abrupt change in the distributed load

(a) Discontinuity in Loading

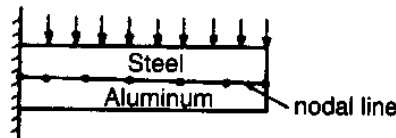


Abrupt change in the cross section of a beam



A cracked plate under loading

(b) Discontinuity in Geometry



A bimetallic beam

(c) Discontinuity in Material Properties

2. DISPLACEMENT OR INTERPOLATION FUNCTION:

The basic principle of FEM is to sub – divide the solution domain into number of finite elements and representing the solution within each element by relatively simple function. This simple function is called '*displacement function or interpolation function*'. The polynomial function is commonly used for this displacement model in FEM due the following two reasons.

1. The polynomials are easier for the mathematical operation i.e. the differentiation and integration of polynomials are easy as compared to trigonometric functions.

2. A polynomial of arbitrary order permits us to have a required approximation i.e by selecting the order of polynomial we can vary the degree of approximation.

The degree of approximation is mainly depends on the order of the polynomial selected for the displacement model as shown in Fig. 2. In this Fig.2 an exact solution for the displacement $u(x)$ for a simple one – dimensional problem is approximated by the following degree of polynomials of the form,

$$u(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \dots + \alpha_{n+1} x^n \quad \text{---(a)}$$

Similarly, for two and three dimensional problems, the polynomial form of interpolation functions can be expressed as

$$u(x, y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2 + \dots + \alpha_m y^n \quad \text{--- (b)}$$

$$u(x, y, z) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 z + \alpha_5 x^2 + \alpha_6 y^2 + \alpha_7 z^2 + \dots + \alpha_m z^n \quad \text{--- (c)}$$

The greater the number of terms included in the polynomial, the solution is more close to the exact solution as shown in Fig. 2(a) to 2(c). The coefficient of the polynomial, the α 's in Eqn. (a) are called *generalized coordinates*.

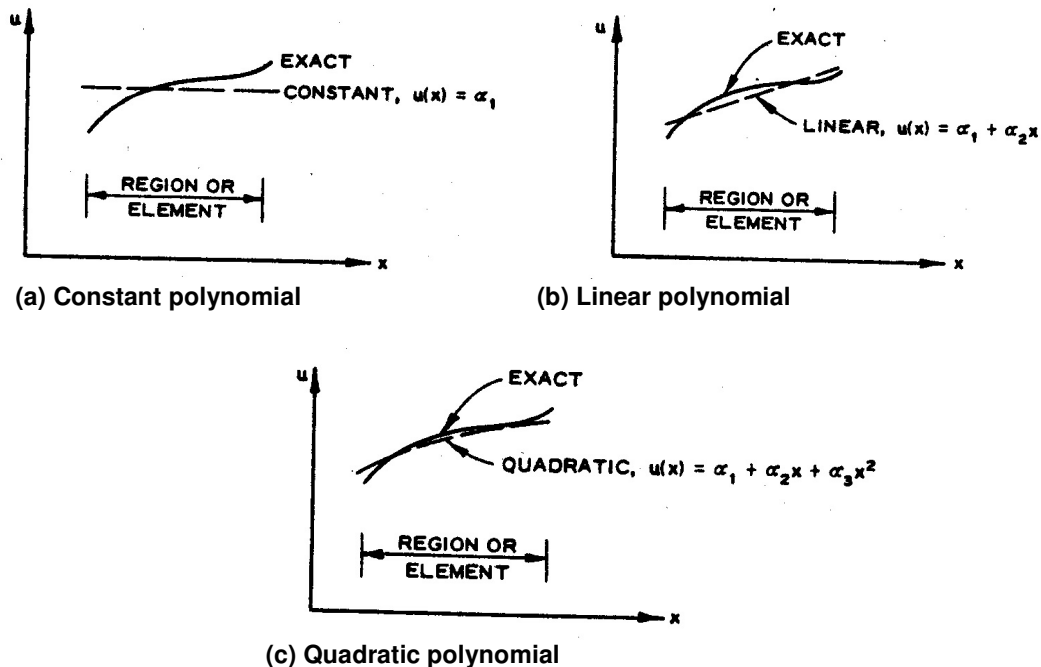


Fig. 2: Polynomial approximation in one – dimension

In most of the cases, the order of the polynomial in interpolation function is taken as one, two or three. Thus, the above equations reduce to the following form for various cases.

For Linear Model (i.e. $n=1$)

For 1D problem: $u(x) = \alpha_1 + \alpha_2 x$

For 2D problem: $u(x, y) = \alpha_1 + \alpha_2 x + \alpha_3 y$

For 3D problem: $u(x, y, z) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 z$

For quadratic Model ($n=2$)

For 1D problem: $u(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2$

For 2D problem: $u(x, y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2$

For 3D problem: $u(x, y, z) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 z + \alpha_5 x^2 + \alpha_6 y^2 + \alpha_7 z^2 + \alpha_8 xy + \alpha_9 yz + \alpha_{10} zx$

3. NODAL DEGREES OF FREEDOM:

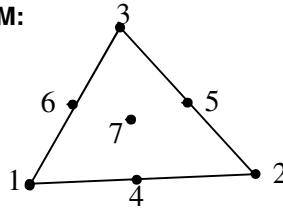


Fig. 4: 7 – noded triangular element

The nodal displacement, rotation and/or strains necessary to specify completely the deformation of the element are referred as *degrees of freedom of the element*. The *nodal degrees of freedom* are the nodal displacement, rotations and/or strains corresponding to the external nodes of an element. On the other hand, the internal degrees of freedom are corresponds to the internal node of an element.

For example, in a triangular element shown in Fig. 4, the nodes 1, 2 and 3 are the primary external nodes and nodes 4, 5 and 6 are secondary external nodes. The node 7 is an internal node. The displacements corresponding to these primary and secondary external nodes 1 to 6 are called *nodal degrees of freedom* and the displacement corresponding to the internal node 7 is called *internal degrees of freedom*.

4 SELECTION OF THE ORDER OF THE INTERPOLATION POLYNOMIAL: While selecting the order of the polynomial in interpolation function, the following considerations have to be taken into account:

1. The interpolation polynomial should satisfy, as far as possible, the convergence requirements.
2. The pattern of variation of field variable resulting from the polynomial model should be independent of local coordinate system.
3. The number of generalized coordinates (α 's) should be equal to the number of nodal degrees of freedom of the element.

5. SIMPLEX, COMPLEX AND MULTIPLEX ELEMENTS: Finite elements can be classified into three categories as simplex, complex, and multiplex elements depending on the geometry of the element and the order of the polynomial used in the interpolation function.

Simplex Elements: The simplex elements are those for which the approximating polynomial consists of constant and linear terms. Thus, the following polynomials represent the simplex functions for one-, two-, and three-dimensional elements.

For 1D problem: $u(x) = \alpha_1 + \alpha_2 x$ --- (a)

For 2D problem: $u(x, y) = \alpha_1 + \alpha_2 x + \alpha_3 y$ --- (b)

For 3D problem: $u(x, y, z) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 z$ --- (c)

The simplex element in one dimension is a 2-noded bar element and for two dimensions is a triangle with three nodes (corners).

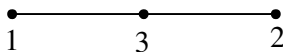
Complex Elements: The complex elements are those for which the approximating polynomial consists of quadratic, cubic, and higher order terms, according to the need, in addition to the constant and linear terms. Thus, the following polynomials denote complex functions.

For 1D problem: $u(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2$ --- (d)

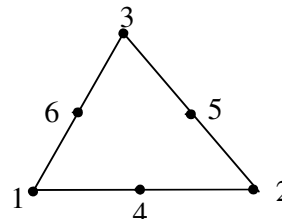
For 2D problem: $u(x, y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2$ --- (e)

For 3D problem: $u(x, y, z) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 z + \alpha_5 x^2 + \alpha_6 y^2 + \alpha_7 z^2 + \alpha_8 xy + \alpha_9 yz + \alpha_{10} zx$

The complex elements may have the same shape as the simplex elements but they have additional external and internal nodes. For example, eq. (e) represents the quadratic interpolation function for a two-dimensional element. Since this equation has 6 α 's, the corresponding element should have six nodes and six-noded triangular element forms a complex element.



(a) For 1 – D problem



(b) For 2 – D problem

Fig (a) Complex Elements

Multiplex Elements: Multiplex elements are those whose boundaries are parallel to the coordinate axes to achieve interelement continuity as shown in Fig. (b) and whose polynomials contain higher order terms.

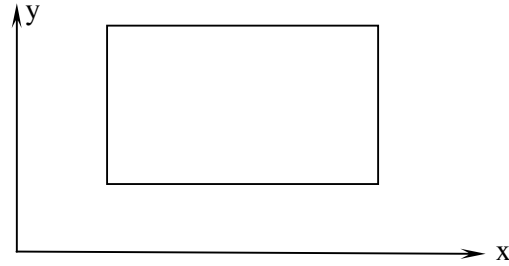
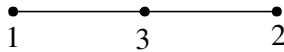


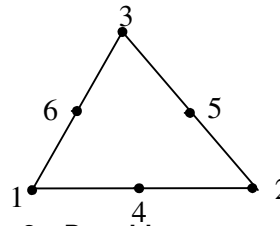
Fig. (b) Multiplex Element

6. Higher Order Elements:

In certain applications, additional degrees of freedom behind the minimum number may be included for any element. This can be achieved by adding secondary external nodes. Then, the elements with these additional degrees of freedom are called as higher order elements. Fig. 5 shows the higher order elements used in 1 – D and 2 – D problems.



(a) For 1 – D problem



(b) For 2 – D problem

Fig. 5: Higher order elements

7. CONVERGENCE REQUIREMENTS: In FEM sequence of approximate solutions are obtained as the element size is reduced successively. These solutions converge to the exact solutions if the interpolation polynomial satisfies the following convergence requirements:

1. *The displacement model must be continuous within the element and the displacement must be compatible between adjacent elements:* The condition can be satisfied by choosing polynomials which are inherently continuous functions for the displacement model. The second condition indicates that when the elements deform there should not be any discontinuity between the elements (elements should not overlap or separate).

2. *The displacement model must be capable of representing rigid body displacement of the element:* This condition states that when the nodes are given such displacements corresponding to a rigid body motion, the element should not experience any strains and hence leads zero nodal forces. The constant term present in the polynomial functions satisfies this condition.

3. *The displacement model must be capable of representing constant strain state within the element:* When the elements are made smaller and smaller size, the strain in each element approaches constant value. In such case, the assumed displacement model should include the terms to represent the constant strain state within the element. For one, two and three - dimensional elasticity problems, the linear terms (i.e. α_2) present in the polynomial functions satisfy this requirement.

In order to satisfy the above mentioned three requirements, the displacement model must contain at least constant and linear terms in the polynomials.

8. GEOMETRIC INVARIANCE OR GEOMETRIC ISOTROPY:

In addition to the convergence and compatibility requirements, one of the important considerations in choosing proper terms in the polynomial displacement model is that the element has no preferred direction. That is, the displacement shapes should not change with change in local coordinate system. This property of the displacement model is known as *geometric invariance or geometric isotropy*.

Geometric isotropy is achieved if the polynomial selected in the displacement model includes all the terms i.e. if the polynomial is complete one. However, the geometric isotropy can also be achieved if the polynomial is balanced in case all the terms can't be included. This balanced representation can be achieved using 'Pascal Triangle' shown in Fig. 7 for the case of two – dimensional polynomials.

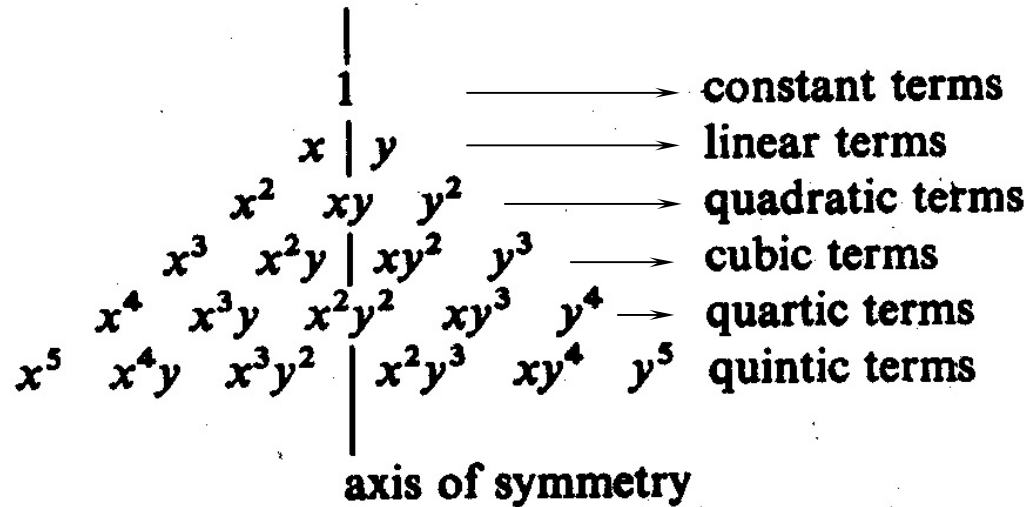


Fig. 7: Pascal triangle for 2 – D problems

For the polynomial functions used in displacement model to be balanced one, it should not include any terms from one side of the axis of the symmetry of the triangle without including its counter part from the other side. For example, if we wish to construct a cubic model with 8 terms, the following are the geometrically isotropic.

1. All the constant, linear and quadratic terms plus x^3 and y^3 .

i.e. $u(x, y) = \alpha_1 + \alpha_2x + \alpha_3y + \alpha_4x^2 + \alpha_5xy + \alpha_6y^2 + \alpha_7x^3 + \alpha_8y^3$

2. All the constant, linear and quadratic terms plus x^2y and xy^2 .

i.e. $u(x, y) = \alpha_1 + \alpha_2x + \alpha_3y + \alpha_4x^2 + \alpha_5xy + \alpha_6y^2 + \alpha_7x^2y + \alpha_8xy^2$

The displacement model formed by any of the above two balanced polynomials will have the geometric isotropy.

9. Coordinate System:

The following coordinate systems are used in FEM.

1. Global coordinate system
2. Local coordinate system
3. Natural coordinate system

Global coordinate system: The coordinate system used to define entire body is called global coordinate system. Fig. 8(a) shows the cartesian global coordinate system used for some of the typical elements.

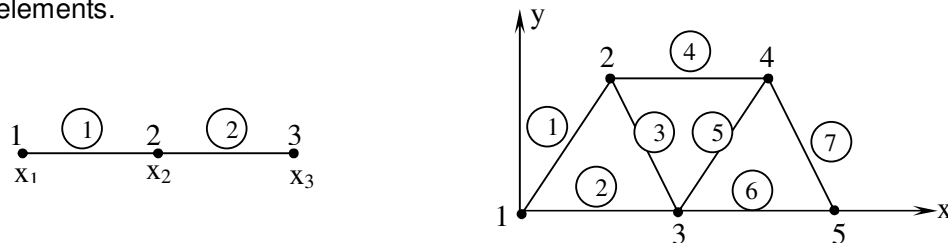


Fig. 8(a): Global coordinate system

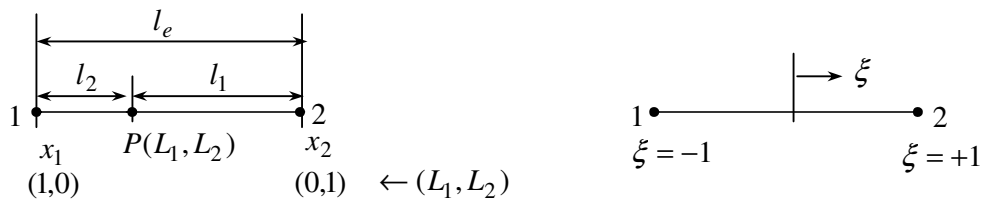
Local coordinate system: A *local coordinate system* is one that is defined for a particular element and not for the entire body. This local coordinate system simplifies the derivation of element formulations. For example, for typical truss elements shown in Fig. 8(a), the local coordinates $(\bar{x}, \bar{y})$ may be as shown in Fig. 8(b). However, for this truss problem the final

equations derived in terms of local coordinate are to be transformed to the common global coordinate system.

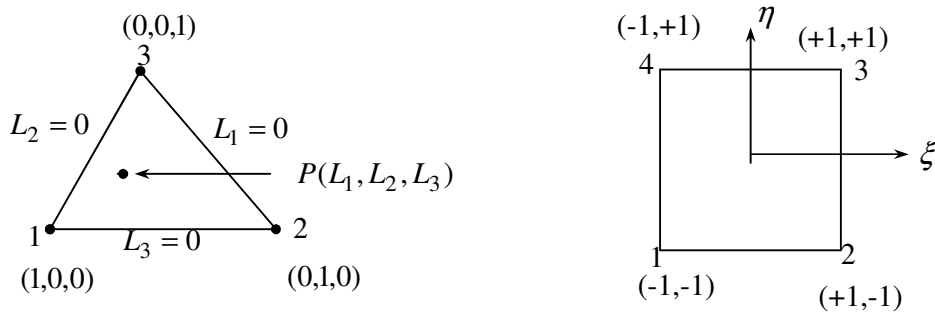
Fig. 8(b): Local coordinate system

Natural Coordinate Systems: The *natural coordinate system* is a local system used to specify the position of a point within the element by a set of dimensionless numbers. The magnitudes of these number never exceed unity. Natural coordinate systems are so defined that, their values are zero or one at the nodal points of an element. This coordinate system facilitates the integration while computing element stiffness matrix.

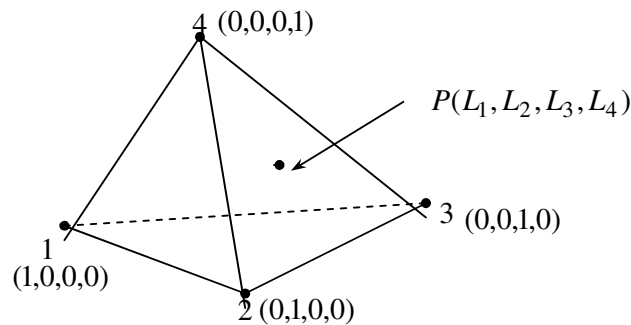
Fig. 8 shows some of the natural coordinate systems used for one, two and three – dimensional elements.



(a) For 1 – D bar elements



(b) For 2 – D elements



(c) For 3 – D element

Fig. 8: Natural coordinate systems

Advantage of natural coordinate system:

1. Natural coordinate systems generalize and simplify the element formulations

2. They simplify the integration and differentiation, which are required to obtain the element stiffness matrix and load vectors.

Area coordinate: If any point $P(x,y)$ divides the total area of the triangular element A into A_1 , A_2 and A_3 as shown in Fig. 9, then areas A_1 , A_2 and A_3 are called *area coordinates*. These are also natural coordinates and these can be used to express the natural coordinates L_1, L_2 & L_3 as

$$L_1 = \frac{A_1}{A}; L_2 = \frac{A_2}{A}; \text{ and } L_3 = \frac{A_3}{A}$$

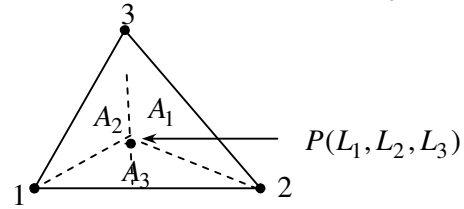


Fig. 9

10. Shape Function or Interpolation Function:

In FEM, it is necessary in many cases to deal with functions whose analytical form is totally unknown. In such case, the given function is replaced by another function which can be more easily handled. This operation of replacing a given function by simple one is known as *Interpolation or shape function*.

Shape function is a function used to interpolate nodal variables to field variables at any point inside the element. Using these shape functions, the field variable can be expressed in terms of nodal variables as

$$u = N_1 u_1 + N_2 u_2 + \dots + N_n u_n$$

where n is the number of nodes in a element.

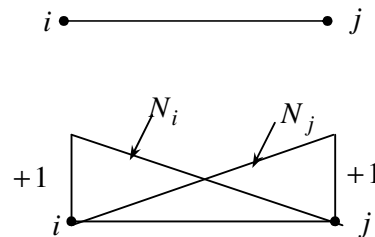
Properties of Shape functions:

The shape functions, which are used to interpolate nodal variable to field variable, are having the following properties

1. The shape functions are having unit value at one nodal point and zero value at all other nodal points. For example, in a 1 – D bar element having nodal points (i, j) , the above mentioned property of shape functions can be written in mathematical form as

$$\text{At node } i, N_i = 1 \quad N_j = 0$$

$$\text{At node } j, N_i = 0 \quad N_j = 1$$



2. The sum of the shape functions is equal to unity

$$\text{i.e. } N_i + N_j = 1$$

3. The derivatives of a shape function is constant

$$\text{i.e. } \frac{dN_i}{dx} = \frac{dN_j}{dx} = \text{Constant}$$

11. Derivation of Expressions for Shape Functions:

(a) For a 2-noded bar element:

(i) In terms of natural coordinate:

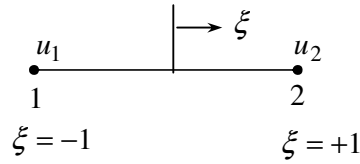


Fig. shows a two noded bar element. The values of natural coordinates corresponding to node 1 and 2 are as shown in Fig. Let the displacement at node 1 and 2 be u_1 and u_2 respectively. Then the field variable (i.e. displacement) u at any point inside the element can be expressed in natural coordinate as

$$u = \alpha_1 + \alpha_2 \xi \quad \text{---(a)}$$

$$u = [1 \quad \xi] \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} \quad \text{---(b)}$$

Substitute the values of ξ corresponding to node 1 and 2 into Eqn. (a), we get nodal displacements

i.e. at node 1, $\xi = -1$ and $u = u_1$

$$\therefore u_1 = \alpha_1 - \alpha_2 \quad \text{---(1)}$$

at node 2, $\xi = +1$ and $u = u_2$

$$\therefore u_2 = \alpha_1 + \alpha_2 \quad \text{---(2)}$$

Add Eqn (1) and (2)

$$u_1 + u_2 = 2\alpha_1$$

$$\therefore \alpha_1 = \frac{u_1}{2} + \frac{u_2}{2} \quad \text{---(i)}$$

Substitute into Eqn (1) we get

$$\alpha_2 = -\frac{u_1}{2} + \frac{u_2}{2} \quad \text{---(ii)}$$

Eqns. (i) and (ii) can be written in matrix form as

$$\begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

Substitute into Eqn. (b)

$$u = [1 \quad \xi] \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$u = \left[\frac{1}{2}(1-\xi) \quad \frac{1}{2}(1+\xi) \right] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$u = [N_1 \quad N_2] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = N_1 u_1 + N_2 u_2 \quad \text{---(c)}$$

where $N_1 = \frac{1}{2}(1 - \xi)$ and $N_2 = \frac{1}{2}(1 + \xi)$ are shape functions for 2 – noded bar element and is interpolate the nodal variables u_1 and u_2 to field variable u as seen from Eqn. (c).

(ii) In terms of Cartesian coordinates

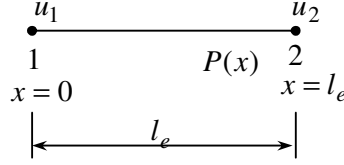


Fig. shows a two noded bar element. The values of cartesian coordinates corresponding to node 1 and 2 are as shown in Fig. Let the displacement at node 1 and 2 be u_1 and u_2 respectively. Then the field variable (i.e. displacement) u at any point inside the element can be expressed in terms of cartesian coordinate as

$$u = \alpha_1 + \alpha_2 x \quad \text{---(a)}$$

$$u = \begin{bmatrix} 1 & x \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} \quad \text{---(b)}$$

Let us consider a coordinate system as shown in Fig. (i.e. at node1, $x = 0$ and at node2 $x = l_e$). Then substitution of the nodal coordinates yields nodal displacements as
i.e. at node 1, $x = 0$ and $u = u_1$

$$\begin{aligned} \therefore u_1 &= \alpha_1 \\ \text{or } \alpha_1 &= u_1 \end{aligned} \quad \text{---(i)}$$

at node 2, $x = l_e$ and $u = u_2$

$$\begin{aligned} \therefore u_2 &= \alpha_1 + \alpha_2 l_e = u_1 + \alpha_2 l_e & \therefore \alpha_1 &= u_1 \\ \therefore \alpha_2 &= -\frac{u_1}{l_e} + \frac{u_2}{l_e} & \text{---(ii)} \end{aligned}$$

Eqns (i) and (ii) can be written in matrix form as

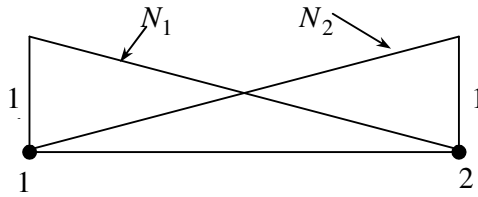
$$\begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{l_e} & \frac{1}{l_e} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \quad \text{---(c)}$$

Substitute in to Eqn.(b), we get

$$u = \begin{bmatrix} 1 & x \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{l_e} & \frac{1}{l_e} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$u = \begin{bmatrix} 1 - \frac{x}{l_e} & \frac{x}{l_e} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = N_1 u_1 + N_2 u_2$$

where, $N_1 = 1 - \frac{x}{l_e}$ and $N_2 = \frac{x}{l_e}$ are shape functions for 2-noded bar element and is interpolate the nodal variables u_1 and u_2 to field variable u as seen from the above equation. The variation of shape functions along the length of element is as shown in Fig. below.



Note: Students are advised to practice the derivation of shape functions either in terms of cartesian or natural coordinate system and to remember the expressions for shape functions in terms of both coordinates.

(b) For a 3-noded bar element (Quadratic bar element):

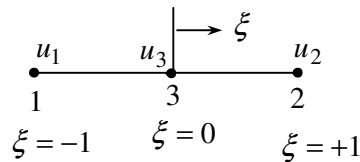


Fig. shows a 1 – D three noded bar element. The values of natural coordinates corresponding to node 1, 2 and 3 are as shown in Fig. Let the displacement at node 1, 2 and 3 be u_1 , u_2 and u_3 respectively. Then the field variable (i.e. displacement) u at any point inside the element can be expressed in natural coordinate as

$$u = \alpha_1 + \alpha_2 \xi + \alpha_3 \xi^2 \quad \text{---(a)}$$

$$u = \begin{bmatrix} 1 & \xi & \xi^2 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} \quad \text{---(b)}$$

Substitute the values of ξ corresponding to node 1, 2 and 3 into Eqn. (a), we get nodal displacement

i.e. at node 1, $\xi = -1$ and $u = u_1$

$$\therefore u_1 = \alpha_1 - \alpha_2 + \alpha_3 \quad \text{---(1)}$$

at node 2, $\xi = +1$ and $u = u_2$

$$\therefore u_2 = \alpha_1 + \alpha_2 + \alpha_3 \quad \text{---(2)}$$

at node 3, $\xi = 0$ and $u = u_3$

$$\therefore u_3 = \alpha_1 \quad \text{---(3)}$$

$$\therefore \alpha_1 = u_3 \quad \text{---(i)}$$

Subtract Eqn (1) from (2)

$$u_2 - u_1 = 2\alpha_2$$

$$\therefore \alpha_2 = -\frac{u_1}{2} + \frac{u_2}{2} \quad \text{---(ii)}$$

Substitute α_1 & α_2 into Eqn (1) we get

$$u_1 = u_3 - \left(-\frac{u_1}{2} + \frac{u_2}{2} \right) + \alpha_3$$

$$\therefore \alpha_3 = \frac{u_1}{2} + \frac{u_2}{2} - u_3 \quad \text{---(iii)}$$

Eqns. (i) to (iii) can be written in matrix form as

$$\begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & -1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

Substituting into Eqn. (b), we get

$$u = \begin{bmatrix} 1 & \xi & \xi^2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & -1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$u = \begin{bmatrix} \left(-\frac{1}{2}\xi + \frac{1}{2}\xi^2\right) & \left(\frac{1}{2}\xi + \frac{1}{2}\xi^2\right) & (1 - \xi^2) \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$u = \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

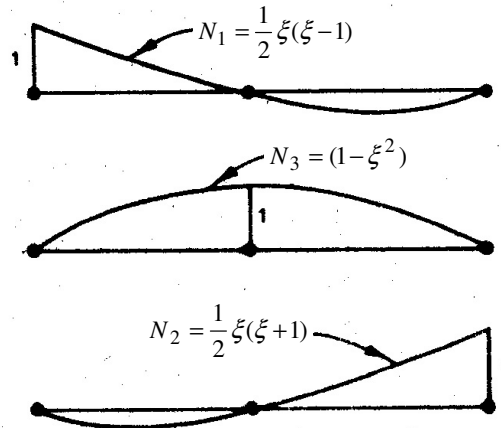
where,

$$N_1 = \left(-\frac{1}{2}\xi + \frac{1}{2}\xi^2\right) = \frac{1}{2}\xi(\xi - 1)$$

$$N_2 = \left(\frac{1}{2}\xi + \frac{1}{2}\xi^2\right) = \frac{1}{2}\xi(\xi + 1)$$

$$N_3 = (1 - \xi^2)$$

are the shape functions for the three noded bar element and are varies over the length of element as shown below.



(c) For a 3-noded triangular element (CST element):

(i) In terms of natural coordinates:

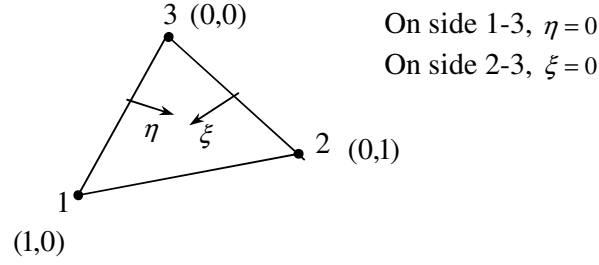


Fig. shows a 3-noded triangular element. The values of natural coordinates corresponding to node 1, 2 and 3 are as shown in Fig. Let the displacement at node 1, 2 and 3 in x-direction be u_1 , u_2 and u_3 respectively. Then the field variable (i.e. displacement) u at any point inside the element can be expressed in natural coordinate as

$$u = \alpha_1 + \alpha_2 \xi + \alpha_3 \eta \quad \text{---(a)}$$

$$u = [1 \quad \xi \quad \eta] \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} \quad \text{---(b)}$$

Substitute the values of ξ and η corresponding to node 1, 2 and 3 into Eqn. (a), we get the nodal displacements

i.e. at node 1, $\xi = 1; \eta = 0$ and $u = u_1$

$$\therefore u_1 = \alpha_1 + \alpha_2 \quad \text{---(1)}$$

at node 2, $\xi = 0; \eta = 1$ and $u = u_2$

$$\therefore u_2 = \alpha_1 + \alpha_3 \quad \text{---(2)}$$

at node 3, $\xi = 0; \eta = 0$ and $u = u_3$

$$\begin{aligned} \therefore u_3 &= \alpha_1 \\ \therefore \alpha_1 &= u_3 \end{aligned} \quad \text{---(i)}$$

Substitute into Eqn. (1), we get

$$\begin{aligned} u_1 &= u_3 + \alpha_2 \\ \therefore \alpha_2 &= u_1 - u_3 \end{aligned} \quad \text{---(ii)}$$

Similarly, substitute into Eqn. (2), we get

$$\alpha_3 = u_2 - u_3 \quad \text{---(iii)}$$

Eqns. (i) to (iii) can be written in matrix form as

$$\begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

Substitute into Eqn. (b), we get

$$\begin{aligned} u &= [1 \quad \xi \quad \eta] \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = [\xi \quad \eta \quad (1-\xi-\eta)] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \\ u &= [N_1 \quad N_2 \quad N_3] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \end{aligned}$$

where, $N_1 = \xi$; $N_2 = \eta$ and $N_3 = 1 - \xi - \eta$ are the shape functions for a three noded triangular (CST) element.

(ii) In terms of cartesian coordinate:

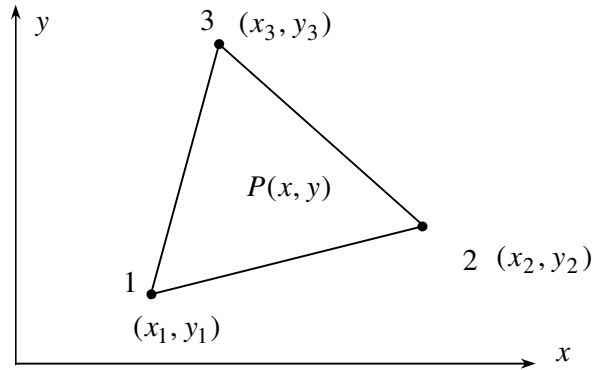


Fig. shows a 3-noded triangular element. The values of natural coordinates corresponding to node 1, 2 and 3 are as shown in Fig. Let the displacement at node 1, 2 and 3 in x-direction be u_1 , u_2 and u_3 respectively. Then the field variable (i.e. displacement) u at any point inside the element can be expressed in terms of cartesian coordinate as

$$u = \alpha_1 + \alpha_2 x + \alpha_3 y \quad \text{---(a)}$$

$$u = \begin{bmatrix} 1 & x & y \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} \quad \text{---(b)}$$

Substitute the values of x and y correspond to node 1, 2 and 3 into Eqn. (a), we get the nodal displacements

i.e. at node 1, $x = x_1$; $y = y_1$ and $u = u_1$

$$\therefore u_1 = \alpha_1 + \alpha_2 x_1 + \alpha_3 y_1 \quad \text{---(1)}$$

at node 2, $x = x_2$; $y = y_2$ and $u = u_2$

$$\therefore u_2 = \alpha_1 + \alpha_2 x_2 + \alpha_3 y_2 \quad \text{---(2)}$$

at node 3, $x = x_3$; $y = y_3$ and $u = u_3$

$$\therefore u_3 = \alpha_1 + \alpha_2 x_3 + \alpha_3 y_3 \quad \text{---(3)}$$

Eqns. (1) – (3) can be written in matrix form as

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix}$$

or $\{q\} = [A]\{\alpha\}$

$$\therefore \{\alpha\} = [A]^{-1}\{q\} \quad \text{---(c)}$$

The inversion of matrix A is given by

$$A^{-1} = \frac{1}{2A_e} \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \quad \text{---(d)}$$

$$\text{where, } A_e = \frac{1}{2} \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

$$\begin{aligned} a_1 &= x_2 y_3 - x_3 y_2 & b_1 &= y_2 - y_3 & c_1 &= x_3 - x_2 \\ a_2 &= x_3 y_1 - x_1 y_3 & b_2 &= y_3 - y_1 & c_2 &= x_1 - x_3 \\ a_3 &= x_1 y_2 - x_2 y_1 & b_3 &= y_1 - y_2 & c_3 &= x_2 - x_1 \end{aligned}$$

where, 1,2,3 are in cyclic order.

Note:

$$a_1 = M_{11} = (-1)^{1+1} \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} = +1(x_2 y_3 - x_3 y_2)$$

$$b_1 = M_{12} = (-1)^{1+2} \begin{vmatrix} 1 & y_2 \\ 1 & y_3 \end{vmatrix} = -1(y_3 - y_2) = y_2 - y_3$$

$$c_1 = M_{13} = (-1)^{1+3} \begin{vmatrix} 1 & x_2 \\ 1 & x_3 \end{vmatrix} = x_3 - x_2$$

$$a_1 = \begin{vmatrix} 1 & -x_1 & -y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

$$b_1 = \begin{vmatrix} 1 & -x_1 & -y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

$$c_1 = \begin{vmatrix} 1 & -x_1 & -y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

Substitute Eqn. (d) into (c), we get

$$\begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} = \frac{1}{2A_e} \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \quad \text{---(e)}$$

substitute Eqn. (e) into Eqn. (b), we get

$$u = [1 \quad x \quad y] \frac{1}{2A_e} \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$u = \left[\frac{1}{2A_e} (a_1 + b_1 x + c_1 y) \quad \frac{1}{2A_e} (a_2 + b_2 x + c_2 y) \quad \frac{1}{2A_e} (a_3 + b_3 x + c_3 y) \right] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$u = [N_1 \quad N_2 \quad N_3] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = N_1 u_1 + N_2 u_2 + N_3 u_3$$

$$\text{where shape functions } N_i = \frac{1}{2A_e} (a_i + b_i x + c_i y) \quad (i = 1, 2, 3)$$

(d) For 4-noded Tetrahedron Element:

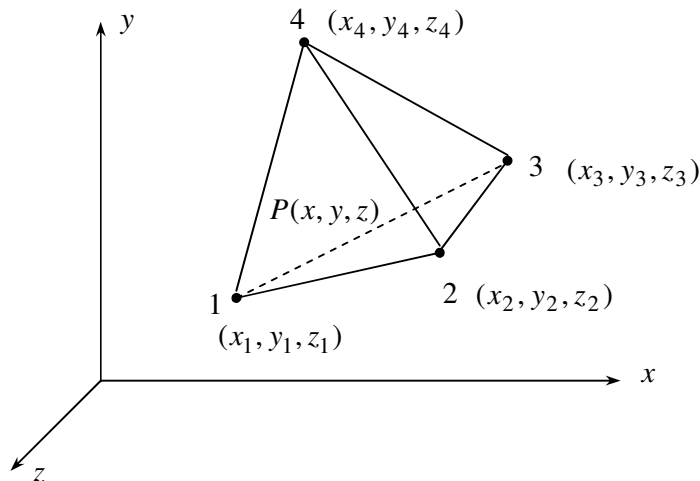


Fig. shows a 4-noded tetrahedron element. The values of natural coordinates corresponding to node 1, 2, 3 and 4 are as shown in Fig. Let the displacement at node 1, 2, 3 and 4 in x-direction be u_1 , u_2 , u_3 and u_4 respectively. Then the field variable (i.e. displacement) u at any point inside the element can be expressed in terms of cartesian coordinate as

$$u = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 z \quad \text{---(a)}$$

$$u = \begin{bmatrix} 1 & x & y & z \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix} \quad \text{---(b)}$$

Substitute the values of x and y correspond to node 1, 2 and 3 into Eqn. (a), we get the nodal displacements

$$u_1 = \alpha_1 + \alpha_2 x_1 + \alpha_3 y_1 + \alpha_4 z_1 \quad \text{---(1)}$$

$$u_2 = \alpha_1 + \alpha_2 x_2 + \alpha_3 y_2 + \alpha_4 z_2 \quad \text{---(2)}$$

$$u_3 = \alpha_1 + \alpha_2 x_3 + \alpha_3 y_3 + \alpha_4 z_3 \quad \text{---(3)}$$

$$u_4 = \alpha_1 + \alpha_2 x_4 + \alpha_3 y_4 + \alpha_4 z_4 \quad \text{---(4)}$$

Eqns. (1) – (4) can be written in matrix form as

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \\ 1 & x_4 & y_4 & z_4 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix}$$

$$\text{or } \{q\} = [A]\{\alpha\}$$

$$\therefore \{\alpha\} = [A]^{-1}\{q\} \quad \text{---(c)}$$

The inversion of matrix A is given by

$$A^{-1} = \frac{1}{6V_e} \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{bmatrix} \quad \text{---(d)}$$

$$\text{where, } V_e = \frac{1}{6} \begin{vmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \\ 1 & x_4 & y_4 & z_4 \end{vmatrix}; \text{ Volume of element}$$

In general,

$$a_1 = M_{11} = +1 \begin{vmatrix} x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{vmatrix}; \quad b_1 = M_{12} = -1 \begin{vmatrix} 1 & y_2 & z_2 \\ 1 & y_3 & z_3 \\ 1 & y_4 & z_4 \end{vmatrix};$$

$$c_1 = M_{13} = +1 \begin{vmatrix} 1 & x_2 & z_2 \\ 1 & x_3 & z_3 \\ 1 & x_4 & z_4 \end{vmatrix}; \quad d_1 = M_{14} = -1 \begin{vmatrix} 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \\ 1 & x_4 & y_4 \end{vmatrix}$$

where, 1,2,3,4 are in cyclic order.

Substitute Eqn. (d) into (c), we get

$$\begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix} = \frac{1}{6V_e} \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} \quad \text{---(e)}$$

Substitute Eqn. (e) into Eqn. (b), we get

$$u = [1 \quad x \quad y \quad z] \frac{1}{6V_e} \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

$$u = [N_1 \quad N_2 \quad N_3 \quad N_4] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = N_1 u_1 + N_2 u_2 + N_3 u_3 + N_4 u_4$$

where shape functions $N_i = \frac{1}{6V_e} (a_i + b_i x + c_i y + d_i z) \quad (i = 1, 2, 3, 4)$

(e) For a 4-noded quadrilateral element:

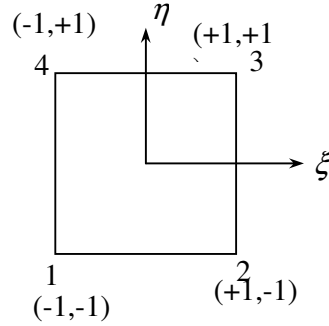


Fig. shows a 2 – D, 4 – noded quadrilateral element. The values of natural coordinates corresponding to node 1, 2, 3 and 4 are as shown in Fig. Let the displacement at node 1, 2, 3 and 4 in x-direction be u_1 , u_2 , u_3 and u_4 respectively. Then the field variable (i.e. displacement) u at any point inside the element can be expressed in natural coordinate as

$$u = \alpha_1 + \alpha_2 \xi + \alpha_3 \eta + \alpha_4 \xi \eta \quad \text{---(a)}$$

$$u = [1 \quad \xi \quad \eta \quad \xi \eta] \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix} \quad \text{---(b)}$$

Substitute the values of ξ and η corresponding to node 1, 2, 3 and 4 into Eqn. (a), we get the nodal displacements

i.e. at node 1, $\xi = -1; \eta = -1$ and $u = u_1$

$$\therefore u_1 = \alpha_1 - \alpha_2 - \alpha_3 + \alpha_4 \quad \text{---(1)}$$

at node 2, $\xi = +1; \eta = -1$ and $u = u_2$

$$\therefore u_2 = \alpha_1 + \alpha_2 - \alpha_3 - \alpha_4 \quad \text{---(2)}$$

at node 3, $\xi = +1; \eta = +1$ and $u = u_3$

$$\therefore u_3 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \quad \text{---(3)}$$

at node 4, $\xi = -1; \eta = +1$ and $u = u_4$

$$\therefore u_3 = \alpha_1 - \alpha_2 + \alpha_3 - \alpha_4 \quad \text{---(4)}$$

Add Eqns. (1) and (2)

$$u_1 + u_2 = 2\alpha_1 - 2\alpha_3 \quad \text{---(5)}$$

$$\therefore 2\alpha_3 = 2\alpha_1 - u_1 - u_2 \quad \text{---(6)}$$

Add Eqns. (3) and (4)

$$u_3 + u_4 = 2\alpha_1 + 2\alpha_3$$

Substituting for $2\alpha_3$, we get

$$u_3 + u_4 = 2\alpha_1 + 2\alpha_1 - u_1 - u_2 = 4\alpha_1 - u_1 - u_2$$

$$\therefore \alpha_1 = \frac{1}{4}(u_1 + u_2 + u_3 + u_4) \quad \text{---(i)}$$

Substitute α_1 into Eqn. (6), we get

$$2\alpha_3 = 2 * \frac{1}{4}(u_1 + u_2 + u_3 + u_4) - u_1 - u_2 = \frac{2u_1 + 2u_2 + 2u_3 + 2u_4 - 4u_1 - 4u_2}{4}$$

$$\alpha_3 = \frac{1}{4}(-u_1 - u_2 + u_3 + u_4) \quad \text{---(iii)}$$

Subtract Eqn. (1) from Eqn. (2)

$$u_2 - u_1 = 2\alpha_2 - 2\alpha_4$$

$$\therefore 2\alpha_4 = 2\alpha_2 + u_1 - u_2 \quad \text{---(7)}$$

Subtract Eqn. (3) from Eqn. (4)

$$u_4 - u_3 = -2\alpha_2 - 2\alpha_4$$

Substitute for $2\alpha_4$ from Eqn. (7), we get

$$u_4 - u_3 = -2\alpha_2 - (2\alpha_2 + u_1 - u_2) = -4\alpha_2 - u_1 + u_2$$

$$\therefore \alpha_2 = \frac{1}{4}(-u_1 + u_2 + u_3 - u_4) \quad \text{---(ii)}$$

Substitute into Eqn. (7), we get

$$\therefore 2\alpha_4 = 2 * \frac{1}{4}(-u_1 + u_2 + u_3 - u_4) + u_1 - u_2$$

$$\therefore \alpha_4 = \frac{1}{4}(u_1 - u_2 + u_3 - u_4) \quad \text{---(iv)}$$

Eqns. (i) to (iv) can be written in matrix form as

$$\begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

Substitute into Eqn. (b), we get

$$u = [1 \quad \xi \quad \eta \quad \xi\eta] \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

$$u = \begin{bmatrix} \frac{1}{4}(1-\xi-\eta+\xi\eta) & \frac{1}{4}(1+\xi-\eta-\xi\eta) & \frac{1}{4}(1+\xi+\eta+\xi\eta) & \frac{1}{4}(1-\xi+\eta-\xi\eta) \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

$$u = [N_1 \quad N_2 \quad N_3 \quad N_4] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

where, $N_1 = \frac{1}{4}(1-\xi-\eta+\xi\eta) = \frac{1}{4}(1-\xi)(1-\eta)$

$$N_2 = \frac{1}{4}(1+\xi)(1-\eta)$$

$$N_3 = \frac{1}{4}(1+\xi)(1+\eta)$$

$$N_4 = \frac{1}{4}(1-\xi)(1+\eta)$$

Note:

In general, the above shape functions for a 4-noded quadrilateral element can be written as

$$N_i = \frac{1}{4}(1+\xi\xi_i)(1+\eta\eta_i); \quad i = 1,2,3,4$$

where, ξ_i and η_i are the values of natural coordinates corresponding to node ($i = 1,2,3,4$).

For example, at node 1 (i.e. $i = 1$), $\xi_1 = -1, \eta_1 = -1$ then shape function

$$N_1 = \frac{1}{4}(1+\xi(-1))(1+\eta(-1)) = \frac{1}{4}(1-\xi)(1-\eta)$$

Similarly, at node 2 (i.e. $i = 2$), $\xi_2 = +1, \eta_2 = -1$ then shape function

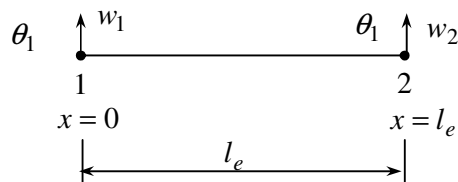
$$N_2 = \frac{1}{4}(1+\xi)(1-\eta) \text{ so on.}$$

For a 2-noded bar element, the general expression for shape functions can also be written as

$$N_i = \frac{1}{2}(1+\xi\xi_i); \quad i = 1,2$$

at node 1, $\xi_1 = -1$, therefore $N_1 = \frac{1}{2}(1-\xi)$ and at node 2, $\xi_2 = +1$, therefore $N_2 = \frac{1}{2}(1+\xi)$ as we obtained in section (a).

(f) For a 2-noded beam element:



Consider a typical 2-noded beam element as shown in Fig. At each node, the unknowns are

displacement w and slop θ and hence there are 2 d.o.f/node and 4 d.o.f/element. Therefore, the deformation of beam element at any point can be expressed as

$$w = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3 \quad \text{---(a)}$$

and slop is given by differentiating w w.r.t x

$$\theta = \frac{dw}{dx} = \alpha_2 + 2\alpha_3 x + 3\alpha_4 x^2 \quad \text{---(b)}$$

Eqn. (a) can also be written in matrix form as

$$w = \begin{bmatrix} 1 & x & x^2 & x^3 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix} \quad \text{---(c)}$$

Let us consider the local coordinate system as shown in Fig. (i.e. at node 1, $x = 0$ and at node 2, $x = l_e$). Then, substitution of nodal coordinates yields nodal deformations and slops as

$$w_1 = \alpha_1 + \alpha_2 0 + \alpha_3 0 + \alpha_4 0 = \alpha_1 \quad \text{---(i)}$$

$$\theta_1 = \alpha_2 + 2\alpha_3 0 + 3\alpha_4 0 = \alpha_2 \quad \text{---(ii)}$$

$$w_2 = \alpha_1 + \alpha_2 l_e + \alpha_3 l_e^2 + \alpha_4 l_e^3 \quad \text{---(iii)}$$

$$\theta_2 = \alpha_2 + 2\alpha_3 l_e + 3\alpha_4 l_e^2 \quad \text{---(iv)}$$

or in matrix form

$$\begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & l_e & l_e^2 & l_e^3 \\ 0 & 1 & 2l_e & 3l_e^2 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix}$$

$$\{d\} = [A]\{\alpha\}$$

$$\therefore \{\alpha\} = [A]^{-1}\{d\} \quad \text{---(d)}$$

where A^{-1} is given by (see note)

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{3}{l_e^2} & -\frac{2}{l_e} & \frac{3}{l_e^2} & -\frac{1}{l_e} \\ \frac{2}{l_e^3} & \frac{1}{l_e^2} & \frac{-2}{l_e^3} & \frac{1}{l_e^2} \end{bmatrix}$$

Substitution of A^{-1} into Eqn.(d) yields

$$\begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{3}{l_e^2} & -\frac{2}{l_e} & \frac{3}{l_e^2} & -\frac{1}{l_e} \\ \frac{2}{l_e^3} & \frac{1}{l_e^2} & \frac{-2}{l_e^3} & \frac{1}{l_e^2} \end{bmatrix} \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix}$$

Substitute above Eqn. into Eqn. (c), we get

$$w = \begin{bmatrix} 1 & x & x^2 & x^3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{3}{l_e^2} & -\frac{2}{l_e} & \frac{3}{l_e^2} & -\frac{1}{l_e} \\ \frac{2}{l_e^3} & \frac{1}{l_e^2} & -\frac{2}{l_e^3} & \frac{1}{l_e^2} \end{bmatrix} \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix}$$

$$w = \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix} \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix} = N_1 w_1 + N_2 \theta_1 + N_3 w_2 + N_4 \theta_2$$

where,

$$N_1 = 1 - \frac{3x^2}{l_e^2} + \frac{2x^3}{l_e^3}; \quad N_2 = x - \frac{2x^2}{l_e} + \frac{x^3}{l_e^2}$$

$$N_3 = \frac{3x^2}{l_e^2} - \frac{2x^3}{l_e^3}; \quad N_4 = -\frac{x^2}{l_e} + \frac{x^3}{l_e^2}$$

Note: The A^{-1} matrix can be written from the shape function expression knowing that elements in 1st, 2nd, 3rd and 4th row are respectively the coefficients of x^0, x^1, x^2 and x^3 terms of shape functions.

$$i.e. A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{3}{l_e^2} & -\frac{2}{l_e} & \frac{3}{l_e^2} & -\frac{1}{l_e} \\ \frac{2}{l_e^3} & \frac{1}{l_e^2} & -\frac{2}{l_e^3} & \frac{1}{l_e^2} \end{bmatrix} \begin{matrix} \leftarrow \text{Coefficients of } x^0 \text{ term of shape functions} \\ \leftarrow \text{Coefficients of } x^1 \text{ term of shape functions} \\ \leftarrow \text{Coefficients of } x^2 \text{ term of shape functions} \\ \leftarrow \text{Coefficients of } x^3 \text{ term of shape functions} \end{matrix}$$

12. Lagrange Interpolation Function or Lagrange polynomials:

Lagrange interpolation function is used to derive shape functions for any line elements and the concept of line element can be easily extended to higher order elements such as 9-noded rectangular element. For a line having n nodes, the Lagrange interpolation function associated with node ' i ' is defined as

$$L_i(x) = \prod_{\substack{m=1 \\ m \neq i}}^n \frac{x - x_m}{x_i - x_m} = \frac{(x - x_1)(x - x_2) \dots (x - x_n)}{(x_i - x_1)(x_i - x_2) \dots (x_i - x_n)}$$

The Lagrange function satisfy the following property

$$L_i(x_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

where x_j denotes x coordinate of j^{th} node. The shape function of a line element is defined as $N_i = L_i(x)$.

The Lagrange interpolation function of a line element can be extended to two dimensional

rectangular elements by taking the tensor product of the x - direction one – dimensional interpolation functions, $L_i(x)$ with the y - direction one dimensional interpolation functions, $L_i(y)$.

i.e. $N_i(x, y) = L_i(x)L_i(y)$

12.1 Derivation of shape functions using Lagrange interpolation function:

(a) 2-noded bar element:

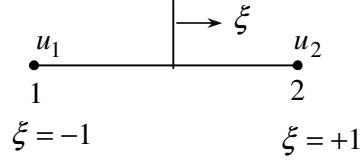


Fig. shows a two noded bar element. The values of natural coordinates corresponding to node 1 and 2 are as shown in Fig. The shape functions N_1 and N_2 for a 2-noded bar element can be derived using Lagrange interpolation function as follows.

The Lagrange interpolation function at node 1 (i.e. $i = 1$) can be written in terms of natural coordinate ξ as

$$L_1(\xi) = \frac{\xi - \xi_2}{\xi_1 - \xi_2} = \frac{\xi - (+1)}{-1 - 1} = -\frac{\xi - 1}{2} = \frac{1}{2}(1 - \xi)$$

Similarly for node 2 (i.e. $i = 2$), it can be written as

$$L_2(\xi) = \frac{\xi - \xi_1}{\xi_2 - \xi_1} = \frac{\xi - (-1)}{1 - (-1)} = \frac{\xi + 1}{2} = \frac{1}{2}(1 + \xi)$$

Then, shape function N_i is related with L_i as

$$N_1 = L_1(\xi) = \frac{1}{2}(1 - \xi) \text{ and } N_2 = L_2(\xi) = \frac{1}{2}(1 + \xi)$$

(b) For a 3-noded bar element:

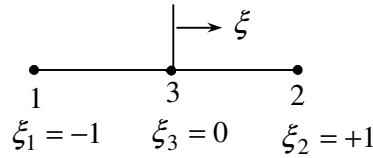


Fig. shows a 3-noded bar element. The values of natural coordinates corresponding to node 1, 2 and 3 are as shown in Fig. The shape functions N_1 , N_2 and N_3 for a 3-node bar element can be derived using Lagrange interpolation function as follows.

The Lagrange interpolation function at node 1 (i.e. $i = 1$) can be written in terms of natural coordinate ξ as

$$L_1(\xi) = \frac{(\xi - \xi_2)(\xi - \xi_3)}{(\xi_1 - \xi_2)(\xi_1 - \xi_3)} = \frac{(\xi - (+1))(\xi - 0)}{(-1 - 1)(-1 - 0)} = \frac{1}{2}\xi(\xi - 1)$$

Similarly for node 2 and 3 (i.e. $i = 2$ and 3), it can be written as

$$L_2(\xi) = \frac{(\xi - \xi_1)(\xi - \xi_3)}{(\xi_2 - \xi_1)(\xi_2 - \xi_3)} = \frac{(\xi + 1)(\xi - 0)}{(1 - (-1))(1 - 0)} = \frac{1}{2}\xi(\xi + 1)$$

$$L_3(\xi) = \frac{(\xi - \xi_1)(\xi - \xi_2)}{(\xi_3 - \xi_1)(\xi_3 - \xi_2)} = \frac{(\xi + 1)(\xi - 1)}{(0 + 1)(0 - 1)} = -(\xi^2 - 1) = (1 - \xi^2)$$

Then, shape function N_i is related with L_i as

$$N_1 = L_1(\xi) = \frac{1}{2}\xi(\xi-1); N_2 = L_2(\xi) = \frac{1}{2}\xi(\xi+1); N_3 = 1-\xi^2$$

(c) For a 4-noded quadrilateral element:

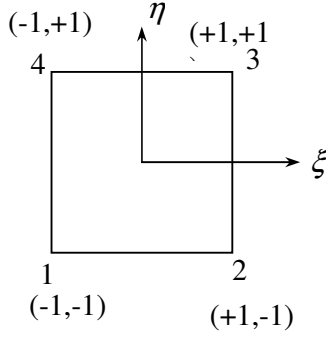


Fig. (a)

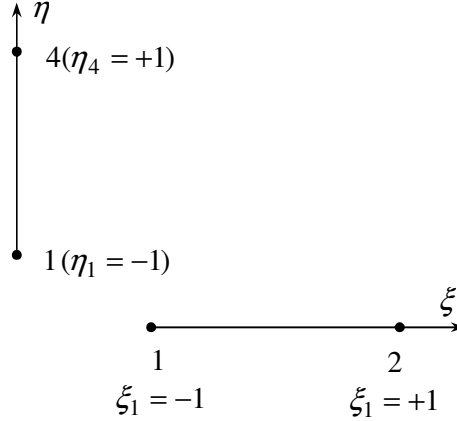


Fig. (b)

Fig. (a) shows a 4 – noded quadrilateral element. The values of natural coordinates corresponding to node 1, 2, 3 and 4 are as shown in Fig. The shape functions for a two dimensional 4-noded quadrilateral element can be derived by taking the tensor product of the ξ - direction one – dimensional interpolation functions, $L_i(\xi)$ with the η - direction one dimensional interpolation functions, $L_i(\eta)$. For any i^{th} node it is written as

$$\therefore N_i = L_i(\xi)L_i(\eta) \quad \text{---(a)}$$

where $L_i(\xi)$ and $L_i(\eta)$ are obtained as follows.

The Lagrange interpolation function for node 1 and 2 in ξ – direction is obtained by considering side 1-2 of quadrilateral element (Fig. (b)) as 1-D 2-noded bar element and is obtained as (Refer case (a))

$$L_1(\xi) = \frac{\xi - \xi_2}{\xi_1 - \xi_2} = \frac{\xi - (+1)}{-1 - 1} = -\frac{\xi - 1}{2} = \frac{1}{2}(1 - \xi)$$

$$L_2(\xi) = \frac{\xi - \xi_1}{\xi_2 - \xi_1} = \frac{\xi - (-1)}{1 - (-1)} = \frac{\xi + 1}{2} = \frac{1}{2}(1 + \xi)$$

From Fig. (a) it can be seen that, the value of ξ at node-4 is same as node-1 and that at node-3 is same as node-2. Hence, we can write

$$L_4(\xi) = L_1(\xi) = \frac{1}{2}(1 - \xi) \quad \because \xi_4 = \xi_1 (= -1)$$

$$L_3(\xi) = L_2(\xi) = \frac{1}{2}(1 + \xi) \quad \because \xi_3 = \xi_2 (= +1)$$

For node 1 and 4, the Lagrange interpolation function can be obtained in η – direction as

$$L_1(\eta) = \frac{\eta - \eta_4}{\eta_1 - \eta_4} = \frac{1}{2}(1 - \eta)$$

$$L_4(\eta) = \frac{\eta - \eta_1}{\eta_4 - \eta_1} = \frac{1}{2}(1 + \eta)$$

$$L_2(\eta) = L_1(\eta) = \frac{1}{2}(1 - \eta) \quad \because \eta_2 = \eta_1 (= -1)$$

$$L_3(\eta) = L_4(\eta) = \frac{1}{2}(1 + \eta) \quad \because \eta_3 = \eta_4 (= +1)$$

Then, shape functions are given by

$$N_1 = L_1(\xi)L_1(\eta) = \frac{1}{2}(1-\xi)\frac{1}{2}(1-\eta) = \frac{1}{4}(1-\xi)(1-\eta)$$

$$N_2 = L_2(\xi)L_2(\eta) = \frac{1}{2}(1+\xi)\frac{1}{2}(1-\eta) = \frac{1}{4}(1+\xi)(1-\eta)$$

$$N_3 = L_3(\xi)L_3(\eta) = \frac{1}{2}(1+\xi)\frac{1}{2}(1+\eta) = \frac{1}{4}(1+\xi)(1+\eta)$$

$$N_4 = L_3(\xi)L_3(\eta) = \frac{1}{2}(1-\xi)\frac{1}{2}(1+\eta) = \frac{1}{4}(1-\xi)(1+\eta)$$

(d) For a 9-noded quadrilateral element:

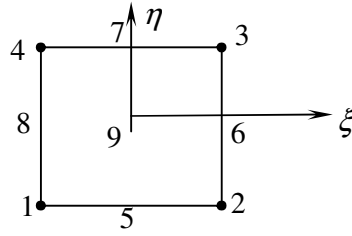


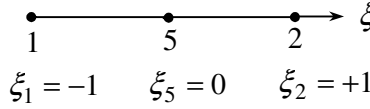
Fig. (b)

The Lagrange interpolation functions for nodes 1,2 and 5 in ξ –direction (Fig. b) can be obtained by considering side 1-2 as 3-noded bar element and are given by (Refer case (b)).

$$L_1(\xi) = \frac{1}{2}\xi(\xi-1)$$

$$L_2(\xi) = \frac{1}{2}\xi(\xi+1)$$

$$L_5(\xi) = (1-\xi^2)$$

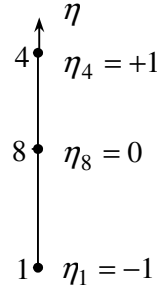


Similarly, Lagrange interpolation functions for nodes 1,8 and 4 in η –direction (Fig. b) can be obtained by considering side 1-4 as 3-noded bar element and are given by

$$L_1(\eta) = \frac{1}{2}\eta(\eta-1)$$

$$L_4(\eta) = \frac{1}{2}\eta(\eta+1)$$

$$L_8(\eta) = (1-\eta^2)$$



Here,

$$L_4(\xi) = L_8(\xi) = L_1(\xi)$$

$$L_3(\xi) = L_6(\xi) = L_2(\xi)$$

$$L_7(\xi) = L_9(\xi) = L_5(\xi)$$

and

$$L_2(\eta) = L_5(\eta) = L_1(\eta)$$

$$L_6(\eta) = L_9(\eta) = L_8(\eta)$$

$$L_3(\eta) = L_7(\eta) = L_4(\eta)$$

Then shape functions are obtained as

$$N_1 = L_1(\xi)L_1(\eta) = \frac{1}{4}\xi\eta(\xi-1)(\eta-1);$$

$$N_2 = L_2(\xi)L_2(\eta) = \frac{1}{4}\xi\eta(\xi+1)(\eta-1)$$

$$\because \xi_4 = \xi_8 = \xi_1$$

$$\because \xi_3 = \xi_6 = \xi_2$$

$$\because \xi_7 = \xi_9 = \xi_5$$

$$\because \eta_2 = \eta_5 = \eta_1$$

$$\because \eta_6 = \eta_9 = \eta_8$$

$$\because \eta_3 = \eta_7 = \eta_4$$

$$\begin{aligned}
N_3 &= L_3(\xi)L_3(\eta) = \frac{1}{4}\xi\eta(\xi+1)(\eta+1); & N_4 &= L_4(\xi)L_4(\eta) = \frac{1}{4}\xi\eta(\xi-1)(\eta+1) \\
N_5 &= L_5(\xi)L_5(\eta) = \frac{1}{2}\eta(1-\xi^2)(\eta-1); & N_6 &= L_6(\xi)L_6(\eta) = \frac{1}{2}\xi(\xi+1)(1-\eta^2) \\
N_7 &= L_7(\xi)L_7(\eta) = \frac{1}{2}\eta(1-\xi^2)(\eta+1); & N_8 &= L_8(\xi)L_8(\eta) = \frac{1}{2}\xi(\xi-1)(1-\eta^2) \\
N_9 &= L_9(\xi)L_9(\eta) = (1-\xi^2)(1-\eta^2)
\end{aligned}$$

13. Lagrange and Serendipity elements:

Lagrange elements:

The shape functions of a 1-D bar element can be easily derived using Lagrange interpolation functions. This approach can be extended to two-dimensional elements (such as 4-noded and 9-noded rectangular elements) by taking tensor product of the x -direction one-dimensional interpolation functions, $L_i(x)$ with the y -direction one-dimensional interpolation function, $L_i(y)$. The above-mentioned procedure can also be extended to derive shape functions for 8-noded and 27-noded three-dimensional elements.

Then, the elements whose shape functions are derived from the tensor product of one-dimensional Lagrange interpolation functions are called Lagrange elements or Lagrange family of elements. Examples of Lagrange elements are two-dimensional 4-noded, 9-noded and 16-noded rectangular elements and three-dimensional 8-noded and 27-noded elements. Fig. 9(a) shows some of typical two-dimensional Lagrange elements.

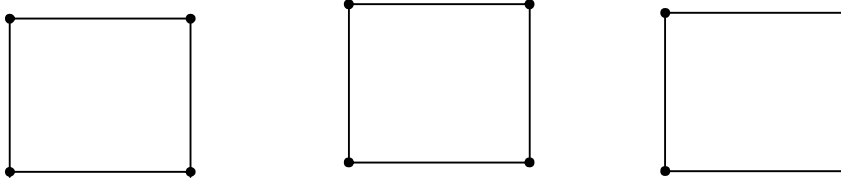


Fig. 9(a): Lagrange elements

Serendipity elements:

The higher order elements, for example 9-noded rectangular element, are having internal node. These nodes do not connect with adjoining elements and hence these nodes can be eliminated at the element level. Alternatively, we can develop elements with nodes present only on the boundaries (i.e. with no internal nodes) and these elements are called serendipity elements. A typical example is the 8-noded or 12-noded rectangular elements as shown in Fig. 9(b).

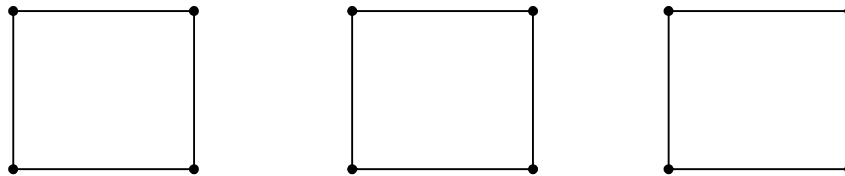


Fig. 9(b): serendipity elements

14. Derivation of Strain-displacement matrix and Stress matrix

(a) For a 2-noded bar element:

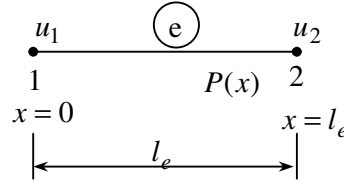


Fig. shows a two noded bar element. The values of cartesian coordinates corresponding to node 1 and 2 are as shown in Fig. Let the displacement at node 1 and 2 be u_1 and u_2 respectively. The strain-displacement, strain and stress matrices can be derived using the following steps.

(1) *Shape functions*: The shape functions for a 2-noded bar element can be expressed as

$$N_1 = 1 - \frac{x}{l_e} \text{ and } N_2 = \frac{x}{l_e} \quad \text{---(a)}$$

(2) *Displacement function or field variable*: the field variable at any point p inside the element can be expressed as

$$u = N_1 u_1 + N_2 u_2 \quad \text{---(b)}$$

$$u = [N_1 \quad N_2] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

(3) *Strain-displacement relation*: for 1-D bar element, the strain-displacement relation can be expressed as

$$\epsilon_x = \frac{\partial u}{\partial x}$$

It may be noted that, u_1 and u_2 in Eqn. (b) are nodal displacement and are constants. Therefore differentiation of Eqn. (b) w.r.t x yields

$$\frac{\partial u}{\partial x} = \frac{\partial N_1}{\partial x} u_1 + \frac{\partial N_2}{\partial x} u_2 = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

Substitute into the above equation, we get strain as

$$\epsilon_x = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\epsilon_x^e = [B]^e \{q\}^e \quad \text{---(c)}$$

Eqn. (c) is known as *Strain matrix*.

Where, $\{q\}^e = \{u_1 \quad u_2\}^T$ is element nodal displacement vector

And $[B]^e = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} \end{bmatrix}$ is *strain-displacement matrix* for a 2-noded element.

Now, differentiate Eqn. (a) w.r.t x , we get

$$\frac{\partial N_1}{\partial x} = -\frac{1}{l_e} \text{ and } \frac{\partial N_2}{\partial x} = \frac{1}{l_e}. \text{ Substitute into } [B]^e \text{ matrix we get the expression for strain-}$$

displacement matrix for a 2-noded element as

$$[B]^e = \begin{bmatrix} -\frac{1}{l_e} & \frac{1}{l_e} \end{bmatrix}$$

It may be noted that, the strain-displacement matrix B is depends only on length of element which is constant. Hence, strain-displacement matrix B for a 2-noded bar element is constant.

(4) *Stress-strain relation*: For a 1-D element, the stress-strain relation can be expressed as

$$\sigma_x^e = E_e \varepsilon_x^e.$$

Substitute for ε_x^e from Eqn. (c), we get

$$\sigma_x^e = E_e [B]^e \{q\}^e \text{ is the stress matrix.}$$

(b) For a 3-noded triangular (C.S.T) element:

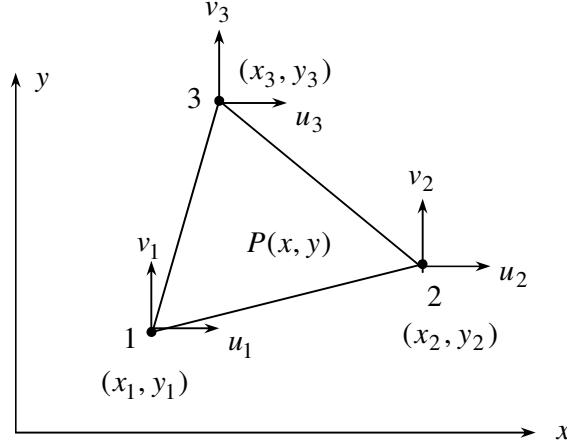


Fig. shows a 3-noded triangular element. The values of cartesian coordinates corresponding to node 1, 2 and 3 are as shown in Fig. Let the displacement at node 1, 2 and 3 in x-direction be u_1 , u_2 and u_3 respectively. The strain-displacement, strain and stress matrices can be derived using the following steps.

(1) *Shape functions*: The shape functions for a 3-noded triangular element can be expressed as

$$\left. \begin{aligned} N_1 &= \frac{1}{2A_e} (a_1 + b_1 x + c_1 y) \\ N_2 &= \frac{1}{2A_e} (a_2 + b_2 x + c_2 y) \\ N_3 &= \frac{1}{2A_e} (a_3 + b_3 x + c_3 y) \end{aligned} \right\} \quad \text{---(a)}$$

where, $a_1 = x_2 y_3 - x_3 y_2$; $b_1 = y_2 - y_3$; $c_1 = x_3 - x_2$ (suffix 1,2,3 are in cyclic order).

(2) *Field variable*: The field variables (i.e. displacements) in x- and y-directions are expressed as

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3$$

$$v = N_1 v_1 + N_2 v_2 + N_3 v_3 \quad \text{---(b)}$$

$$u = \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

(3) *Strain-displacement relations*: for a 2-D problem, the strain-displacement relation is expressed as

$$\varepsilon_x = \frac{\partial u}{\partial x}; \quad \varepsilon_y = \frac{\partial v}{\partial y} \quad \text{and} \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

Differentiate Eqns. (b) w.r.t x and y and substitute into the above equation, we get

$$\varepsilon_x = \frac{\partial u}{\partial x} = \frac{\partial N_1}{\partial x} u_1 + \frac{\partial N_2}{\partial x} u_2 + \frac{\partial N_3}{\partial x} u_3$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = \frac{\partial N_1}{\partial y} v_1 + \frac{\partial N_2}{\partial y} v_2 + \frac{\partial N_3}{\partial y} v_3 \text{ and}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial N_1}{\partial y} u_1 + \frac{\partial N_2}{\partial y} u_2 + \frac{\partial N_3}{\partial y} u_3 + \frac{\partial N_1}{\partial x} v_1 + \frac{\partial N_2}{\partial x} v_2 + \frac{\partial N_3}{\partial x} v_3$$

The above equation can be written in matrix form as

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

$$\{\varepsilon\}^e = [B]^e \{q\}^e \text{ is the strain matrix.} \quad \text{---(c)}$$

$$\text{Where, } [B]^e = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} \end{bmatrix} \text{ is strain-displacement matrix.}$$

In general differentiation of shape functions $N_i (i = 1, 2, 3)$ w.r.t x and y yields

$$\frac{\partial N_i}{\partial x} = \frac{1}{2A_e} b_i \text{ and } \frac{\partial N_i}{\partial y} = \frac{1}{2A_e} c_i. \text{ Hence, substitution of the above derivatives into the above expression yields}$$

$$[B]^e = \frac{1}{2A_e} \begin{bmatrix} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \\ c_1 & b_1 & c_2 & b_2 & c_3 & b_3 \end{bmatrix} \text{ is the strain-displacement matrix for 3-noded triangular}$$

element.

It may be noted that strain-displacement matrix for a 3-noded triangular element is only depends on nodal coordinates which are constant for the given element and hence it is constant. Therefore, 3-noded triangular element is called as *Constant strain Triangular (C.S.T) element*.

(4) *Stress-strain relation*: for 2-D problem, the stress-strain relation is expressed as

$$\{\sigma\}^e = [D]^e \{\varepsilon\}^e, \text{ where, } [D]^e \text{ constitutive relation which relate the strain with stress.}$$

Substitute for $\{\varepsilon\}^e$ from eqn. (c), we get

$$\{\sigma\}^e = [D]^e [B]^e \{q\}^e \text{ is the stress matrix}$$

Note: For 2-D plain stress and plain strain problems the constitutive matrix $[D]$ in the above equation is given by

$$[D] = \frac{E}{(1-\mu^2)} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix} \quad \text{for plain stress problems}$$

$$[D] = \frac{E}{(1+\mu)(1-2\mu)} \begin{bmatrix} 1-\mu & \mu & 0 \\ \mu & 1-\mu & 0 \\ 0 & 0 & \frac{1-2\mu}{2} \end{bmatrix} \quad \text{for plain strain problems}$$

(c) For a 3-noded bar element or 1-D Quadratic element:

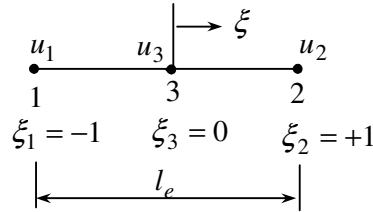


Fig. shows a 3-noded bar element. The values of natural coordinates corresponding to node 1, 2 and 3 are as shown in Fig. The strain-displacement, strain and stress matrices can be derived using the following steps.

(1) *Shape functions*: The shape functions for a 3-noded bar (1-D quadratic) element can be expressed in natural coordinate system as

$$N_1 = \frac{1}{2}\xi(\xi-1), \quad N_2 = \frac{1}{2}\xi(\xi+1) \quad \text{and} \quad N_3 = (1-\xi^2) \quad \text{---(a)}$$

$$\text{where, } \xi = \frac{2(x-x_3)}{l_e} \quad \text{---(b)}$$

(2) *Field variable*: Is expressed as

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3 = \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \quad \text{---(c)}$$

(3) *Strain-displacement relation*: Is expressed as

$$\epsilon_x = \frac{\partial u}{\partial x}. \text{ Differentiate Eqn. (c) w.r.t } x, \text{ we get}$$

$$\epsilon_x = \frac{\partial u}{\partial x} = \frac{\partial N_1}{\partial x} u_1 + \frac{\partial N_2}{\partial x} u_2 + \frac{\partial N_3}{\partial x} u_3$$

$$\epsilon_x^e = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = [B]^e \{q\}^e$$

$$\text{where, } [B]^e = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} \end{bmatrix} \quad \text{---(d)}$$

is strain-displacement matrix.

As the shape functions N_i ($i=1,2,3$) are functions of ξ , their derivatives w.r.t x can be written as

$$\frac{\partial N_i}{\partial x} = \frac{\partial N_i}{\partial \xi} \frac{\partial \xi}{\partial x}.$$

Hence, diff. Eqn. (b) w.r.t x we get

$$\frac{\partial \xi}{\partial x} = \frac{2(1-0)}{l_e} = \frac{2}{l_e}. \text{ Therefore, above equation becomes as}$$

$$\frac{\partial N_i}{\partial x} = \frac{2}{l_e} \frac{\partial N_i}{\partial \xi} \quad (i = 1, 2, 3)$$

Differentiate Eqn.(a) w.r.t ξ and substitute we get

$$\therefore \frac{\partial N_1}{\partial x} = \frac{2}{l_e} \frac{\partial N_1}{\partial \xi} = \frac{2}{l_e} \left(\frac{1}{2} (2\xi - 1) \right)$$

$$\frac{\partial N_2}{\partial x} = \frac{2}{l_e} \frac{\partial N_2}{\partial \xi} = \frac{2}{l_e} \left(\frac{1}{2} (2\xi + 1) \right)$$

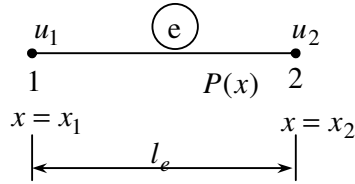
$$\frac{\partial N_3}{\partial x} = \frac{2}{l_e} \frac{\partial N_3}{\partial \xi} = \frac{2}{l_e} (-2\xi)$$

substitute into eqn. (d), we get

$$[B]^e = \frac{2}{l_e} \begin{bmatrix} \frac{1}{2} (2\xi - 1) & \frac{1}{2} (2\xi + 1) & -2\xi \end{bmatrix}$$

15. Derivation of Stiffness matrix and Load Vectors

(a) For a 2-noded bar element:



Consider a 2-noded bar element as shown in Fig. The elemental stiffness matrix and load vectors can be derived from strain energy and work potential of total potential energy functional.

For an e^{th} element, the total potential energy can be written as

$$\pi = \frac{1}{2} \int_V \sigma^T \epsilon dV - \int_V u^T f dV - \int_s u^T T ds \quad \text{---(a)}$$

where, first part of the above equation gives strain energy and second and third parts gives work potential. Using strain energy and work potential, the elemental stiffness and load vectors can be derived for the given element as follows:

(a) Stiffness Matrix:

Consider strain energy functional of total potential energy and is expressed as

$$U_e = \frac{1}{2} \int_{V_e} \sigma^T \epsilon dV_e \quad \text{---(b)}$$

But, the stress strain relation yields

$$\sigma = D\epsilon \quad \text{---(i)}$$

where, matrix D is called constitutive matrix. The strain-displacement relation yields

$$\epsilon = Bq \quad \text{---(ii)}$$

Substitution of ϵ into the above equation yields

$$\sigma = DBq$$

$$\therefore \sigma^T = q^T B^T D \quad (D^T = D \text{ as } D \text{ matrix is symmetric})$$

Substitution of σ^T and ϵ into Eqn. (b) yields

$$U_e = \frac{1}{2} \int_{V_e} (q^T B^T DBq) dV_e = \frac{1}{2} q^T \left(\int_{V_e} B^T DB dV_e \right) q = \frac{1}{2} q^T K^e q$$

where, K_e is *elemental stiffness matrix* and is expressed as

$$K^e = \int_{V_e} B^T DB dV_e \quad \text{---(c)}$$

For 1-D bar element, $D = E_e$ and $dV_e = A_e dx$. Substitute into Eqn.(c), we get

$$K^e = \int_0^{l_e} B^T E_e B A_e dx = A_e E_e B^T B \int_0^{l_e} dx \quad \because \text{matrix } B \text{ is constant matrix}$$

where, B is strain-displacement matrix and for 2-noded bar element it is expressed as

$$B = \begin{bmatrix} -\frac{1}{l_e} & \frac{1}{l_e} \end{bmatrix} \quad \therefore B^T = \begin{bmatrix} -\frac{1}{l_e} \\ \frac{1}{l_e} \end{bmatrix}$$

$$\text{Hence, } B^T B = \begin{bmatrix} -\frac{1}{l_e} & \frac{1}{l_e} \end{bmatrix} \begin{bmatrix} -\frac{1}{l_e} \\ \frac{1}{l_e} \end{bmatrix} = \begin{bmatrix} \frac{1}{l_e^2} & -\frac{1}{l_e^2} \\ -\frac{1}{l_e^2} & \frac{1}{l_e^2} \end{bmatrix}$$

Substitute into K^e and integrate we get

$$K^e = A_e E_e \begin{bmatrix} \frac{1}{l_e^2} & -\frac{1}{l_e^2} \\ -\frac{1}{l_e^2} & \frac{1}{l_e^2} \end{bmatrix} l_e = \frac{A_e E_e}{l_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \text{ is the elemental stiffness matrix for a 2-noded bar}$$

element.

(b) Load vectors: Load vectors due to body force and surface force can be derived from work potential of total potential energy. The work potential for an e^{th} element can be expressed as

$$WP = - \int_{V_e} u^T f dV_e - \int_s u^T T ds \quad \text{---(d)}$$

where, u is field variable (displacement) at any point inside the element. For 2-noded bar element it is expressed as

$$u = N_1 u_1 + N_2 u_2 = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = Nq$$

$$\therefore u^T = q^T N^T \quad \text{---(e)}$$

Substitute Eqn. (e) into Eqn. (d) we get

$$WP = -q^T \int_{V_e} N^T f dV_e - q^T \int_s N^T T ds$$

Here $f = \{f_x \quad f_y \quad f_z\}$ is body force vector and $T = \{T_x \quad T_y \quad T_z\}$ is surface force vector. For an e^{th} 1-D bar element $f = f_x^e$ and $T = T_x^e$ (i.e. simply component of body force and surface force along x-direction and $dV_e = A_e dx$ and $ds = dx$. Therefore, the above equation becomes as

$$WP = -q^T \int_0^{l_e} N^T f_x^e A_e dx - q^T \int_0^{l_e} N^T T_x^e dx \quad \text{---(f)}$$

(i) *Load vector due to body force:* From eqn. (f), the work potential due to body force is

expressed as

$$WP_{b.f} = q^T \left(A_e \int_0^{l_e} N^T f_x^e dx \right) = q^T f^e$$

where, f^e is elemental load vector due to body force and is expressed as

$$f^e = A_e \int_0^{l_e} N^T f_x^e dx = A_e f_x^e \int_0^{l_e} \begin{Bmatrix} N_1 \\ N_2 \end{Bmatrix} dx$$

Using the following integration rule for 2-noded bar element, the above equation can be integrated.

$$\int_0^{l_e} N_1^p N_2^q dx = \frac{p!q!}{(p+q+1)!} l_e$$

$$f^e = A_e f_x^e \int_0^{l_e} \begin{Bmatrix} N_1 \\ N_2 \end{Bmatrix} dx = A_e f_x^e \begin{Bmatrix} \frac{1}{2} l_e \\ \frac{1}{2} l_e \end{Bmatrix}$$

$$\int_0^{l_e} N_1 dx = \frac{1!}{(1+1)!} l_e = \frac{1}{1 \times 2} l_e = \frac{1}{2} l_e$$

$$\int_0^{l_e} N_2 dx = \frac{1!}{(1+1)!} l_e = \frac{1}{1 \times 2} l_e = \frac{1}{2} l_e$$

$$f^e = \frac{A_e l_e f_x^e}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \text{ is the load vector due to body force for the case of 2-noded bar element.}$$

(i) **Load vector due to surface force:** From Eqn. (f), the work potential due to surface force is expressed as

$$WP_{s.f} = q^T \left(\int_0^{l_e} N^T T_x^e dx \right) = q^T T^e$$

where, T^e is elemental load vector due to surface force and is expressed as

$$T^e = \int_0^{l_e} N^T T_x^e dx = T_x^e \int_0^{l_e} \begin{Bmatrix} N_1 \\ N_2 \end{Bmatrix} dx$$

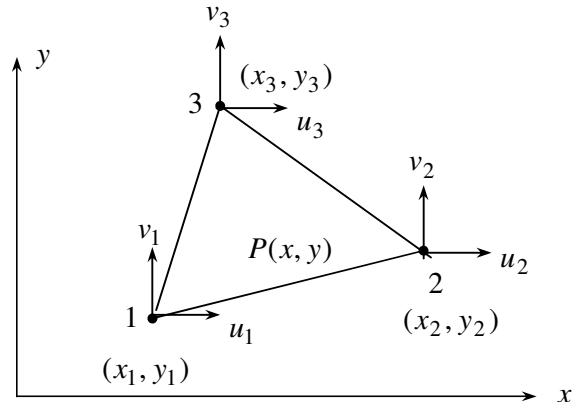
Using the above mentioned integration rule for 2-noded bar element, the above equation can be integrated as

$$T^e = T_x^e \int_0^{l_e} \begin{Bmatrix} N_1 \\ N_2 \end{Bmatrix} dx = T_x^e \begin{Bmatrix} \frac{1}{2} l_e \\ \frac{1}{2} l_e \end{Bmatrix}$$

$$T^e = \frac{l_e T_x^e}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \text{ is the load vector due to surface force for the case of 2-noded bar element.}$$

(b) For a 3-noded triangular (C.S.T) element:

Fig. shows a 3-noded triangular element. The values of cartesian coordinates corresponding to node 1, 2 and 3 are as shown in Fig. The elemental stiffness matrix and load vectors can be derived from strain energy and work potential of total potential energy functional. For an e^{th} element, the total potential energy can be written as



$$\pi = \frac{1}{2} \int_V \sigma^T \varepsilon dV - \int_V u^T f dV - \int_s u^T T ds \quad \text{---(a)}$$

where, first part of the above equation gives strain energy and second and third parts gives work potential. Using strain energy and work potential, the elemental stiffness and load vectors can be derived for the given element as follows:

(a) Stiffness Matrix: Proceed same as case (a) upto Eqn. (c) and show that

$$K^e = \int_{V_e} B^T DB dV_e \quad \text{---(c)}$$

For 2-D element, $dV_e = t_e dA_e$, where t_e is thickness of element. Substitute into Eqn.(c), we get

$$K^e = \int_0^{l_e} B^T DB t_e dA_e = t_e B^T DB \int_{A_e} dA_e$$

$K^e = A_e t_e B^T DB$ is elemental stiffness matrix.

Here, D is constitutive relation for two-dimensional problem and is expressed as

$$D = \frac{E}{(1-\mu^2)} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix} \quad \text{for plain stress problems}$$

$$D = \frac{E}{(1+\mu)(1-2\mu)} \begin{bmatrix} 1-\mu & \mu & 0 \\ \mu & 1-\mu & 0 \\ 0 & 0 & \frac{1-2\mu}{2} \end{bmatrix} \quad \text{for plain strain problems}$$

(b) Load vector: Load vectors due to body force and surface force can be derived from work potential of total potential energy. The work potential for an e^{th} element can be expressed as

$$WP = - \int_{V_e} u^T f dV_e - \int_s u^T T ds \quad \text{---(d)}$$

where, u is field variable (displacement) at any point inside the element.

(i) **Load vector due to body force:** From Eqn. (d), the work potential due to body force is expressed as

$$WP_{b.f} = \int_{V_e} u^T f dV_e \quad \text{---(e)}$$

where, $u = \{u \ v\}^T$ are field variables in x- and y-directions. For 3-noded triangular element, they are expressed as

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3$$

$$v = N_1 v_1 + N_2 v_2 + N_3 v_3$$

or in matrix form

$$u = \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} = [N] \{q\}$$

where, $u = \begin{Bmatrix} u \\ v \end{Bmatrix} = \{u_1 \quad v_1 \quad u_2 \quad v_2 \quad u_3 \quad v_3\}^T$ and $[N] = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix}$
 $\therefore u^T = q^T N^T$

Substitute into Eqn. (e) we get

$$WP_{b,f} = q^T \int_{V_e} N^T f dV_e \quad \text{---(f)}$$

Here $f = \{f_x \quad f_y \quad f_z\}$ is body force vector and for a 2-D element $f = \{f_x \quad f_y\}^T$ and (i.e. the components of body force along x- and y- directions). Then, the above equation becomes as

$$WP_{b,f} = q^T \int_{V_e} \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ N_3 & 0 \\ 0 & N_3 \end{bmatrix} \begin{Bmatrix} f_x \\ f_y \end{Bmatrix} dV_e = q^T f^e \quad \text{---(g)}$$

where, f^e is elemental load vector due to body force and is expressed as

$$f^e = \int_{V_e} \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ N_3 & 0 \\ 0 & N_3 \end{bmatrix} \begin{Bmatrix} f_x \\ f_y \end{Bmatrix} dV_e = t_e \int_{A_e} \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ N_3 & 0 \\ 0 & N_3 \end{bmatrix} \begin{Bmatrix} f_x \\ f_y \end{Bmatrix} dA_e = t_e \int_{A_e} \begin{Bmatrix} N_1 f_x \\ N_1 f_y \\ N_2 f_x \\ N_2 f_y \\ N_3 f_x \\ N_3 f_y \end{Bmatrix} dA_e$$

For 2-D element, $dV_e = t_e dA_e$ where, t_e is thickness of element.

Using the following integration rule for 3-noded triangular element, the above equation can be integrated.

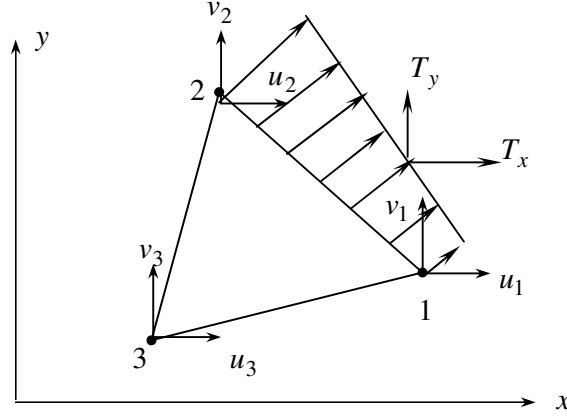
$$\int_{A_e} N_1^p N_2^q N_3^r dA = \frac{p!q!r!}{(p+q+r+2)!} 2A_e$$

$$\text{i.e. } \int_{A_e} N_1^1 dA_e = \int_{A_e} N_2^1 dA_e = \int_{A_e} N_3^1 dA_e = \frac{1!}{(1+2)!} 2A_e = \frac{1}{1 \times 2 \times 3} 2A_e = \frac{A_e}{3}$$

$$f^e = \frac{A_e t_e}{3} \begin{Bmatrix} f_x \\ f_y \\ f_x \\ f_y \\ f_x \\ f_y \end{Bmatrix} \begin{matrix} \swarrow \searrow \\ \swarrow \searrow \\ \swarrow \searrow \end{matrix} \begin{matrix} \text{x- and y- components at node-1} \\ \text{x- and y- components at node-2} \\ \text{x- and y- components at node-3} \end{matrix}$$

is the load vector due to body force for the case of 3-noded triangular element.

(ii) *Load vector due to surface force:* The surface force (traction) is a distributed load acts on the edge of an element connecting the boundary nodes.



Consider a triangular element subjected to uniformly varying load (i.e. traction force, force per unit surface area) on side 1-2 as shown in Fig. The component of this traction force in x- and y- directions are T_x and T_y respectively. Let T_x varies from T_{x1} at node-1 to T_{x2} at node-2 and that T_y from T_{y1} to T_{y2} . From Eqn. (d), the work potential due to body force is expressed as

$$WP_{S.f} = \int_S u^T T dS_e \quad \text{---(e)}$$

where, $u = \{u \ v\}^T$ is a field variable in x- and y-directions. It may be noted that $N_3 = 0$ along side 1-2 and hence side 1-2 can be consider as line element. Then. Field variable between node 1 and 2 is expressed as

$$u = N_1 u_1 + N_2 u_2$$

$$v = N_1 v_1 + N_2 v_2$$

$$u = \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 \\ 0 & N_1 & 0 & N_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix} = Nq$$

$$\therefore u^T = q^T N^T$$

$$\text{where, } q = \{u_1 \ v_1 \ u_2 \ v_2\}^T \text{ and } N = \begin{bmatrix} N_1 & 0 & N_2 & 0 \\ 0 & N_1 & 0 & N_2 \end{bmatrix}$$

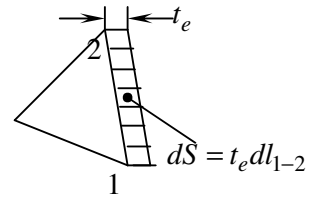
$$WP_{S.f} = q^T \int_S N^T T dS_e \quad \text{---(f)}$$

Further, the traction force at any point on side 1-2 can be expressed in terms of nodal forces as

$$T_x = N_1 T_{x1} + N_2 T_{x2}$$

$$T_y = N_1 T_{y1} + N_2 T_{y2}$$

$$T = \begin{Bmatrix} T_x \\ T_y \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 \\ 0 & N_1 & 0 & N_2 \end{bmatrix} \begin{Bmatrix} T_{x1} \\ T_{y1} \\ T_{x2} \\ T_{y2} \end{Bmatrix}$$



Substituting N^T and T into Eqn.(f), we get

$$WP_{S,f} = q^T t_e \int_{l_{1-2}} \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \end{bmatrix} \begin{bmatrix} N_1 & 0 & N_2 & 0 \\ 0 & N_1 & 0 & N_2 \end{bmatrix} \begin{Bmatrix} T_{x1} \\ T_{y1} \\ T_{x2} \\ T_{y2} \end{Bmatrix} dl = q^T T^e$$

where, T^e is elemental load vector due to surface force and is expressed as

$$T^e = t_e \int_{l_{1-2}} \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \end{bmatrix} \begin{bmatrix} N_1 & 0 & N_2 & 0 \\ 0 & N_1 & 0 & N_2 \end{bmatrix} \begin{Bmatrix} T_{x1} \\ T_{y1} \\ T_{x2} \\ T_{y2} \end{Bmatrix} dl$$

$$T^e = t_e \int_{l_{1-2}} \begin{bmatrix} N_1^2 & 0 & N_1 N_2 & 0 \\ 0 & N_1^2 & 0 & N_1 N_2 \\ N_1 N_2 & 0 & N_2^2 & 0 \\ 0 & N_1 N_2 & 0 & N_2^2 \end{bmatrix} \begin{Bmatrix} T_{x1} \\ T_{y1} \\ T_{x2} \\ T_{y2} \end{Bmatrix} dl$$

Using the following integration rule for line element (because side 1-2 is considered as line), the above equation can be integrated.

$$\int_{l_{1-2}} N_1^p N_2^q dl = \frac{p!q!}{(p+q+1)!} l_{1-2}$$

$$\text{i.e. } \int_{l_{1-2}} N_1^2 dl = \int_{l_{1-2}} N_1^2 dl = \frac{2!}{(2+1)!} l_{1-2} = \frac{l_{1-2}}{3} \text{ and } \int_{l_{1-2}} N_1^1 N_2^1 dl = \frac{1!1!}{(1+1+1)!} l_{1-2} = \frac{l_{1-2}}{6}$$

$$T^e = t_e \begin{bmatrix} \frac{l_{1-2}}{3} & 0 & \frac{l_{1-2}}{6} & 0 \\ 0 & \frac{l_{1-2}}{3} & 0 & \frac{l_{1-2}}{6} \\ \frac{l_{1-2}}{6} & 0 & \frac{l_{1-2}}{3} & 0 \\ 0 & \frac{l_{1-2}}{6} & 0 & \frac{l_{1-2}}{3} \end{bmatrix} \begin{Bmatrix} T_{x1} \\ T_{y1} \\ T_{x2} \\ T_{y2} \end{Bmatrix}$$

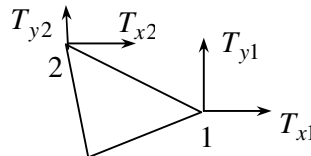
$$T^e = \frac{t_e l_{1-2}}{6} \begin{Bmatrix} 2T_{x1} + T_{x2} \\ 2T_{y1} + T_{y2} \\ T_{x1} + 2T_{x2} \\ T_{y1} + 2T_{y2} \end{Bmatrix} \quad \text{---(g)}$$

is the load vector due to surface force for the case of 3-noded triangular element.

Note:

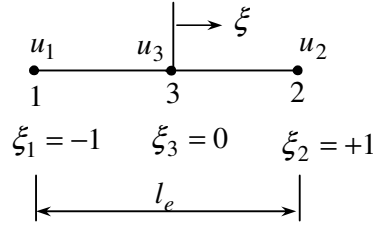
1. If traction force is u.d.l, $T_{x1} = T_{x2}$ and $T_{y1} = T_{y2}$, then Eqn. (g) reduces to

$$T^e = \frac{t_e l_{1-2}}{2} \begin{Bmatrix} T_{x1} \\ T_{y1} \\ T_{x2} \\ T_{y2} \end{Bmatrix}$$



2. If side 2-3 of triangular element is subjected to traction force, then l_{1-2} is replaced by l_{2-3} and suffices 1 & 2 of T_x and T_y are replaced by 2 & 3 respectively in Eqn. (g).

(c) For a 3-noded bar or 1-D Quadratic element:



Consider a quadratic element having 3 nodes as shown in Fig. The values of natural coordinates at nodal points are as shown in Fig. The elemental stiffness matrix and load vectors can be derived from strain energy and work potential of total potential energy functional. For an e^{th} element, the total potential energy can be written as

$$\pi = \frac{1}{2} \int_V \sigma^T \epsilon dV - \int_V u^T f dV - \int_s u^T T ds \quad \text{---(a)}$$

Using strain energy and work potential, the elemental stiffness and load vectors can be derived for the given element as follows:

(a) Stiffness Matrix: Proceed same as case (a) and show that

$$U_e = \frac{1}{2} q^T \left(A_e E_e \int_0^{l_e} B^T B dx \right) q$$

$$U_e = \frac{1}{2} q^T K^e q$$

where $K^e = A_e E_e \int_0^{l_e} B^T B dx$ is a stiffness matrix

Note: Strain-displacement matrix B for a 3-noded bar element is not constant and hence it cannot take out side the integration. Further, matrix B is a function of ξ and hence the above integration w.r.t x should be modified to integrate w.r.t ξ .

$$\text{We have, } \xi = \frac{2(x - x_3)}{l_e} \quad \therefore \frac{d\xi}{dx} = \frac{2}{l_e} \quad \text{or } dx = \frac{l_e}{2} d\xi$$

Substitute into K^e , we get

$$K^e = A_e E_e \frac{l_e}{2} \int_{-1}^{+1} B^T B d\xi \quad \text{---(b)}$$

(limits $0 - l_e$ in the above Eqn. changes to -1 to $+1$)

Here,

$$B = \frac{2}{l_e} \begin{bmatrix} \frac{1}{2}(2\xi - 1) & \frac{1}{2}(2\xi + 1) & -2\xi \end{bmatrix} \therefore B^T B = \frac{4}{l_e^2} \begin{bmatrix} \frac{1}{2}(2\xi - 1) \\ \frac{1}{2}(2\xi + 1) \\ -2\xi \end{bmatrix} \begin{bmatrix} \frac{1}{2}(2\xi - 1) & \frac{1}{2}(2\xi + 1) & -2\xi \end{bmatrix}$$

$$B^T B = \frac{4}{l_e^2} \begin{bmatrix} \frac{1}{4}(2\xi-1)^2 & \frac{1}{4}(2\xi-1)(2\xi+1) & -\frac{1}{2}(2\xi-1)2\xi \\ \frac{1}{4}(2\xi+1)(2\xi-1) & \frac{1}{4}(2\xi+1)^2 & -\frac{1}{2}(2\xi+1)2\xi \\ -2\xi\frac{1}{2}(2\xi-1) & -2\xi\frac{1}{2}(2\xi+1) & 4\xi^2 \end{bmatrix}$$

$$\therefore \int_{-1}^{+1} B^T B d\xi = \frac{4}{l_e^2} \int_{-1}^{+1} \begin{bmatrix} \frac{1}{4}(4\xi^2+1-4\xi) & \frac{1}{4}(4\xi^2-1) & -(2\xi^2-\xi) \\ \text{Symm} & \frac{1}{4}(4\xi^2+1+4\xi) & -(2\xi^2+\xi) \\ & & 4\xi^2 \end{bmatrix} d\xi \quad \text{---(c)}$$

Integration of the following terms yields,

$$\int_{-1}^{+1} \xi^2 d\xi = \left. \frac{\xi^3}{3} \right|_{-1}^{+1} = \frac{1}{3}(1+1) = \frac{2}{3}; \quad \int_{-1}^{+1} \xi d\xi = \left. \frac{\xi^2}{2} \right|_{-1}^{+1} = \frac{1}{2}(1-1) = 0 \quad \text{and} \quad \int_{-1}^{+1} d\xi = \left. \xi \right|_{-1}^{+1} = (1+1) = 2$$

Substitute into Eqn. (c), we get

$$\therefore \int_{-1}^{+1} B^T B d\xi = \frac{4}{l_e^2} \begin{bmatrix} \frac{1}{4}(4 \times \frac{2}{3} + 2 - 4 \times 0) & \frac{1}{4}(4 \times \frac{2}{3} - 2) & -(2 \times \frac{2}{3} - 0) \\ \text{Symm} & \frac{1}{4}(4 \times \frac{2}{3} + 2 + 4 \times 0) & -(2 \times \frac{2}{3}) \\ & & 4 \times \frac{2}{3} \end{bmatrix}$$

$$\int_{-1}^{+1} B^T B d\xi = \frac{1}{3l_e^2} \begin{bmatrix} 14 & 2 & -16 \\ \text{Symm} & 14 & -16 \\ & & 32 \end{bmatrix}$$

Substitute into Eqn. (b), we get

$$K^e = \frac{A_e E_e l_e}{2} \times \frac{1}{3l_e^2} \begin{bmatrix} 14 & 2 & -16 \\ \text{Symm} & 14 & -16 \\ & & 32 \end{bmatrix}$$

$$K^e = \frac{A_e E_e}{3l_e} \begin{bmatrix} 7 & 1 & -8 \\ \text{Symm} & 7 & -8 \\ & & 16 \end{bmatrix} = \frac{A_e E_e}{3l_e} \begin{bmatrix} 7 & 1 & -8 \\ 1 & 7 & -8 \\ -8 & -8 & 16 \end{bmatrix}$$

(b) Load Vector: Load vectors due to body force and surface force can be derived from work potential of total potential energy. The work potential for an e^{th} element can be expressed as

$$WP = - \int_{V_e} u^T f dV_e - \int_s u^T T ds \quad \text{---(d)}$$

where, u is field variable (displacement) at any point inside the element and is expressed as

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3 = \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = Nq$$

$$\therefore u^T = q^T N^T$$

For a bar element, $f = f_x^e$, $T = T_x^e$, $dV_e = A_e dx$ and $ds = dx$,

Substitute into Eqn. (d), we get

$$WP = - \int_0^{l_e} q^T N^T f_x^e A_e dx - \int_0^{l_e} q^T N^T T_x^e dx$$

$$WP = - \int_0^{l_e} q^T N^T f_x^e A_e \frac{l_e}{2} d\xi - \int_0^{l_e} q^T N^T T_x^e \frac{l_e}{2} d\xi \quad \therefore dx = \frac{l_e}{2} \quad \text{---(e)}$$

(i) *Load vector due to body force:* From Eqn. (e), the work potential due to body force is expressed as

$$WP_{b.f} = q^T \frac{A_e l_e f_x^e}{2} \int_0^{l_e} N^T d\xi = q^T f^e$$

where f^e is elemental load vector due to body force and is expressed as

$$f^e = \frac{A_e l_e f_x^e}{2} \int_{-1}^{+1} N^T d\xi = \frac{A_e l_e f_x^e}{2} \int_{-1}^{+1} \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix} d\xi$$

For a 3-node bar element, $N_1 = \frac{1}{2} \xi(\xi-1) = \frac{1}{2}(\xi^2 + \xi)$; $N_2 = \frac{1}{2} \xi(\xi+1) = \frac{1}{2}(\xi^2 - \xi)$ and $N_3 = 1 - \xi^2$. Substitute into above equation we get

$$f^e = \frac{A_e l_e f_x^e}{2} \int_{-1}^{+1} \begin{Bmatrix} \frac{1}{2}(\xi^2 + \xi) \\ \frac{1}{2}(\xi^2 - \xi) \\ 1 - \xi^2 \end{Bmatrix} d\xi \quad \text{---(f)}$$

Integration of following terms yields

$$\int_{-1}^{+1} \xi^2 d\xi = \left. \frac{\xi^3}{3} \right|_{-1}^{+1} = \frac{1}{3}(1+1) = \frac{2}{3}; \quad \int_{-1}^{+1} \xi d\xi = \left. \frac{\xi^2}{2} \right|_{-1}^{+1} = \frac{1}{2}(1-1) = 0 \quad \text{and} \quad \int_{-1}^{+1} d\xi = \xi \Big|_{-1}^{+1} = (1+1) = 2$$

Substitute into Eqn. (f), we get

$$f^e = \frac{A_e l_e f_x^e}{2} \begin{Bmatrix} \frac{1}{2} \left(\frac{2}{3} - 0 \right) \\ \frac{1}{2} \left(\frac{2}{3} + 0 \right) \\ 2 - \frac{2}{3} \end{Bmatrix}. \text{ Finally, the load vector due to body force is expressed as}$$

$$f^e = Ael_e f_x^e \begin{Bmatrix} \frac{1}{6} \\ \frac{1}{6} \\ \frac{2}{3} \end{Bmatrix} \begin{matrix} \leftarrow \text{Load at node-1} \\ \leftarrow \text{Load at node-2} \\ \leftarrow \text{Load at node-3} \end{matrix}$$

(ii) **Load vector due to surface force:** From Eqn. (e), the work potential due to surface force is expressed as

$$WP_{s.f} = q^T \frac{l_e T_x^e}{2} \int_0^{l_e} N^T d\xi = q^T T^e$$

where T^e is elemental load vector due to surface force and is expressed as

$$T^e = \frac{l_e T_x^e}{2} \int_{-1}^{+1} N^T d\xi = \frac{l_e T_x^e}{2} \int_{-1}^{+1} \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix} d\xi = \frac{l_e T_x^e}{2} \int_{-1}^{+1} \begin{Bmatrix} \frac{1}{2}(\xi^2 - \xi) \\ \frac{1}{2}(\xi^2 + \xi) \\ 1 - \xi^2 \end{Bmatrix} d\xi$$

Substitution of integration terms discussed above yields

$$T^e = \frac{l_e T_x^e}{2} \begin{Bmatrix} \frac{1}{2} \times \frac{2}{3} \\ \frac{1}{2} \times \frac{2}{3} \\ 2 - \frac{2}{3} \end{Bmatrix}. \text{ Finally, the load vector due to surface force is expressed as}$$

$$T^e = l_e T_x^e \begin{Bmatrix} \frac{1}{6} \\ \frac{1}{6} \\ \frac{2}{3} \end{Bmatrix} \begin{matrix} \leftarrow \text{Load at node-1} \\ \leftarrow \text{Load at node-2} \\ \leftarrow \text{Load at node-3} \end{matrix}$$

16. Assembly of the Global stiffness matrix and Load vector

The potential energy for an e^{th} element can be expressed as

$$\pi^e = \int_{V_e} \sigma^T \epsilon dV_e - \int_{V_e} u^T f dV_e - \int_{S_e} u^T T dS_e$$

where, the 1st term in RHS of the above equation represents strain energy due to internal stresses and 2nd and 3rd terms represent the work potential due to body force, and surface force respectively. Also, $\epsilon = Bq$; $\sigma^T = q^T B^T D$ and $u^T = q^T N^T$.

Substitution of the above terms into the above equation yields

$$\pi^e = \frac{1}{2} q^T \left(\int_{V_e} B^T D B dV_e \right) q - q^T \left(\int_{V_e} N^T f dV_e \right) - q^T \left(\int_{S_e} N^T T dS_e \right)$$

$$\pi^e = \frac{1}{2} q^T K^e q - q^T f^e - q^T T^e$$

where, K^e = element stiffness matrix

q = element nodal displacement vector

f^e and T^e = element load vectors due to body force and surface force

The total potential energy of the entire body, which is divided into n_e number of elements, is obtained by the summation of the potential energy of each element.

$$\pi = \sum_e \frac{1}{2} q^T \left(\int_{V_e} B^T DB dV_e \right) q - \sum_e q^T \left(\int_{V_e} N^T f dV_e \right) - \sum_e q^T \left(\int_{S_e} N^T T dS_e \right)$$

After performing assembly process over each element, the above equation can be expressed in short form as

$$\pi = \frac{1}{2} Q^T K Q - Q^T F$$

where, K = Global stiffness matrix

Q = global nodal displacement vector

F = Global force vector that includes global load vectors due to body force, surface force and point loads (if any).

The global stiffness matrix K and load vector F are respectively obtained by the assembly of element stiffness matrices and element load vectors using nodal connectivity table.

Prob: Fig.(a) shows a stepped bar. For each element 'i' the cross sectional area and length are A_i and L_i respectively. Each element is subjected to a traction force T_i per unit length and a body force f_i per unit volume. Young's modulus of the material is E and a concentrated load P_2 is applied at node 2. Obtain the global stiffness matrix and nodal load vector.

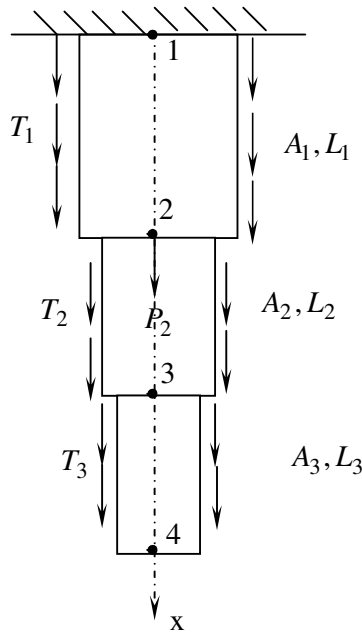


Fig. (a)

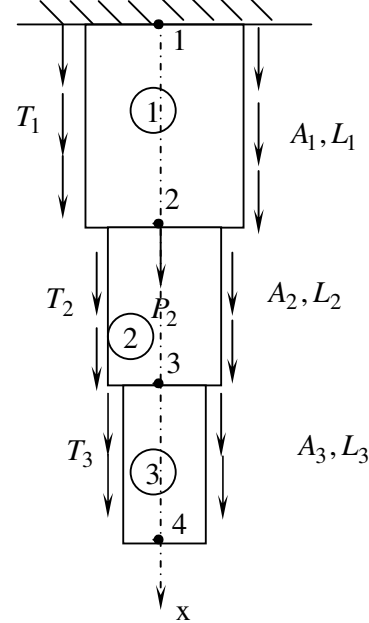


Fig. (b)

Solution: The given stepped bars are discretized into 3 elements as shown in Fig. (b).

(a) To obtain Global Stiffness Matrix:

In general element stiffness matrix is expressed as

$$K^e = \frac{A_e E_e}{l_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Then, for elements 1,2 and 3 it is expressed as

$$K^1 = E \begin{bmatrix} \frac{A_1}{l_1} & -\frac{A_1}{l_1} \\ -\frac{A_1}{l_1} & \frac{A_1}{l_1} \end{bmatrix} \begin{matrix} 1 \\ 2 \end{matrix} \quad K^2 = E \begin{bmatrix} \frac{A_2}{l_2} & -\frac{A_2}{l_2} \\ -\frac{A_2}{l_2} & \frac{A_2}{l_2} \end{bmatrix} \begin{matrix} 2 \\ 3 \end{matrix} \quad \text{and} \quad K^3 = E \begin{bmatrix} \frac{A_3}{l_3} & -\frac{A_3}{l_3} \\ -\frac{A_3}{l_3} & \frac{A_3}{l_3} \end{bmatrix} \begin{matrix} 3 \\ 4 \end{matrix} \quad \text{---(a)}$$

Connectivity table:

Element Nos.	Local node Nos.	Global D.O.F
1	1	1
	2	2
2	1	2
	2	3
3	1	3
	2	4

Write the global D.O.F to the element matrices and assemble the stiffness matrix, we get

$$K = E \begin{bmatrix} \frac{A_1}{l_1} & -\frac{A_1}{l_1} & 0 & 0 \\ -\frac{A_1}{l_1} & \frac{A_1}{l_1} + \frac{A_2}{l_2} & -\frac{A_2}{l_2} & 0 \\ 0 & -\frac{A_2}{l_2} & \frac{A_2}{l_2} + \frac{A_3}{l_3} & -\frac{A_3}{l_3} \\ 0 & 0 & -\frac{A_3}{l_3} & \frac{A_3}{l_3} \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} \quad \text{is the global stiff. Matrix.}$$

(b) To obtain global load vector:

(i) Load vector due to body force:

The elemental load vector due to body force is expressed as, $f^e = \frac{A_e l_e f_x^e}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$

$$\text{For element 1: } f_x^1 = f_1 \quad \therefore f^1 = \frac{A_1 l_1 f_1}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} \frac{A_1 l_1 f_1}{2} \\ \frac{A_1 l_1 f_1}{2} \end{Bmatrix} \begin{matrix} 1 \\ 2 \end{matrix}$$

$$\text{For element 2: } f_x^2 = f_2 \quad \therefore f^2 = \begin{Bmatrix} \frac{A_2 l_2 f_2}{2} \\ \frac{A_2 l_2 f_2}{2} \end{Bmatrix} \begin{matrix} 2 \\ 3 \end{matrix}$$

$$\text{For element 1: } f_x^3 = f_3 \quad \therefore f^3 = \begin{Bmatrix} \frac{A_3 l_3 f_3}{2} \\ \frac{A_3 l_3 f_3}{2} \end{Bmatrix} \begin{matrix} 3 \\ 4 \end{matrix}$$

Write the global d.o.f from connectivity table as shown above and assemble, we get global load vector due to body force as

$$F_{b,f} = \begin{Bmatrix} \frac{A_1 l_1 f_1}{2} \\ \frac{A_1 l_1 f_1}{2} + \frac{A_2 l_2 f_2}{2} \\ \frac{A_2 l_2 f_2}{2} + \frac{A_3 l_3 f_3}{2} \\ \frac{A_3 l_3 f_3}{2} \end{Bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix}$$

(ii) Load vector due to surface force:

The elemental load vector due to surface force is expressed as, $T^e = \frac{l_e T_x^e}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$

$$\text{For element 1: } T_x^1 = T_1 \quad \therefore T^1 = \begin{Bmatrix} \frac{l_1 T_1}{2} \\ \frac{l_1 T_1}{2} \end{Bmatrix} \begin{matrix} 1 \\ 2 \end{matrix}$$

$$\text{For element 2: } T_x^2 = T_2 \quad \therefore T^2 = \begin{Bmatrix} \frac{l_2 T_2}{2} \\ \frac{l_2 T_2}{2} \end{Bmatrix} \begin{matrix} 2 \\ 3 \end{matrix}$$

$$\text{For element 1: } T_x^3 = T_3 \quad \therefore T^3 = \begin{Bmatrix} \frac{l_3 T_3}{2} \\ \frac{l_3 T_3}{2} \end{Bmatrix} \begin{matrix} 3 \\ 4 \end{matrix}$$

Write the global d.o.f from connectivity table as shown above and assemble, we get global load vector due to surface force as

$$F_{s,f} = \begin{Bmatrix} \frac{l_1 T_1}{2} \\ \frac{l_1 T_1}{2} + \frac{l_2 T_2}{2} \\ \frac{l_2 T_2}{2} + \frac{l_3 T_3}{2} \\ \frac{l_3 T_3}{2} \end{Bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix}$$

(iii) Load vector due to point load: The global load vector due to point load is simply obtained by writing the point loads at the global nodes and is given by

$$F_{P,L} = \begin{Bmatrix} R_1 \\ P_2 \\ 0 \\ 0 \end{Bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix}; \text{ where } R_1 \text{ is the reaction at the support.}$$

(iv) Total global load vector:

The total global load vector is obtained by summation of global load vectors due to body, surface and point loads.

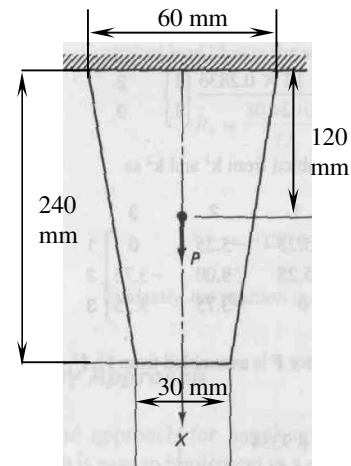
$$F = F_{b.f} + F_{s.f} + F_{P.L} = \begin{Bmatrix} \frac{A_1 l_1 f_1}{2} + \frac{l_1 T_1}{2} + R_1 \\ \frac{A_1 l_1 f_1}{2} + \frac{A_2 l_2 f_2}{2} + \frac{l_1 T_1}{2} + \frac{l_2 T_2}{2} + P_2 \\ \frac{A_2 l_2 f_2}{2} + \frac{A_3 l_3 f_3}{2} + \frac{l_2 T_2}{2} + \frac{l_3 T_3}{2} \\ \frac{A_3 l_3 f_3}{2} + \frac{l_3 T_3}{2} \end{Bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix}$$

16. Application of Boundary Conditions: Refer class notes or Introduction to FE in Engg. By Chandrupatla and Belegundu, pp. 63-75

Problems:

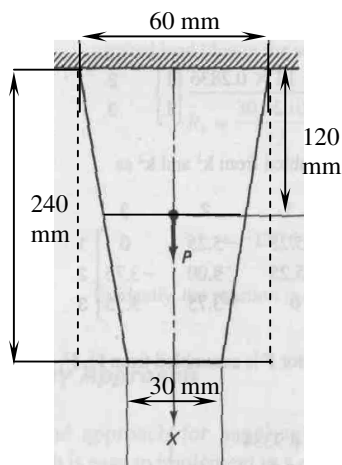
Prob. 1: Fig. shows a thin (steel) plate of uniform thickness of 1mm, Young's modulus $E = 200 \text{ GPa}$ and weight density $\gamma = 76.6 \times 10^{-6} \text{ N/mm}^3$. In addition to its self-weight, the plate is subjected to a point load $P = 100 \text{ N}$ at its midpoint.

- Model the plate with two finite elements.
- Write down expressions for the element stiffness matrices and element body force vectors.
- Assemble the structural stiffness matrix K and global load vector F .
- Using the elimination approach, solve for the global displacement vector Q .
- Evaluate the stresses in each element.
- Determine the reaction force at the support.

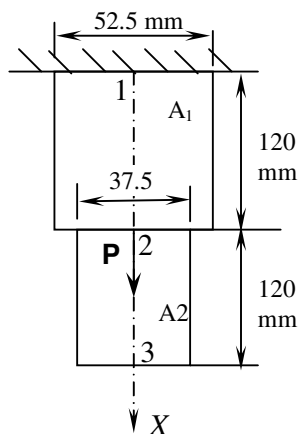


a) To model the plate into two elements:

The given plate is discretized into two elements as shown in Fig. (b) and modeled as two stepped bars by taking average width for each elements as shown in Fig. (c). From Fig. (b), it can be written as



(b)



(c)

$$E_1 = E_2 = 200 \times 10^3 \text{ N/mm}^2$$

$$l_1 = l_2 = 120 \text{ mm}$$

$$f_x^1 = f_x^2 = 76.6 \times 10^{-6} \text{ N/mm}^3$$

$$\tan \theta = \frac{a}{120} = \frac{15}{240} \quad \therefore a = \frac{120 * 15}{240} = 7.5 \text{ mm}$$

Then, width of plate at midpoint is $w_2 = w_1 - 2a = 60 - 2 * 7.5 = 45 \text{ mm}$

$$\text{Therefore, average width of element 1} = \frac{60 + 45}{5} = 52.5 \text{ mm}$$

$$\text{Average width of element 2} = \frac{45 + 30}{2} = 37.5 \text{ mm}$$

Now, C/s area of element 1, $A_1 = 52.5 * 1 = 52.5 \text{ mm}^2$

And C/s area of element 2, $A_2 = 37.5 * 1 = 37.5 \text{ mm}^2$

b) To write expressions for element stiffness matrices and body force vectors:

Element stiffness matrices: In general, the element stiffness matrix for an e^{th} element is expressed as

$$K^e = \frac{A_e E_e}{l_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

For

element

1:

$$K^1 = \frac{A_1 E_1}{l_1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{52.5 * 200 * 10^3}{120} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0.875 * 10^5 & -0.875 * 10^5 \\ -0.875 * 10^5 & 0.875 * 10^5 \end{bmatrix}$$

For element 2:

$$K^2 = \frac{A_2 E_2}{l_2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{37.5 * 200 * 10^3}{120} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0.625 * 10^5 & -0.625 * 10^5 \\ -0.625 * 10^5 & 0.625 * 10^5 \end{bmatrix}$$

Element body force vectors: is expressed as $f^e = \frac{A_e l_e f_x^e}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$

For element 1: here $f_x^1 = 76.6 \times 10^{-6} \text{ N/mm}^3$

$$\therefore f^1 = \frac{52.5 * 120 * 76.6 * 10^{-6}}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0.241 \\ 0.241 \end{Bmatrix}$$

For element 2: $f_x^2 = 76.6 \times 10^{-6} \text{ N/mm}^3$

$$\therefore f^2 = \frac{37.5 * 120 * 76.6 * 10^{-6}}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0.143 \\ 0.143 \end{Bmatrix}$$

c) To assemble stiffness matrix and load vectors:

Stiffness matrix: The assembly of element stiffness matrices can be done by using the following connectivity table:

Element Nos.	Local node Nos.	Global D.O.F
1	1	1
	2	2
2	1	2
	2	3

Write the global D.O.F to the element matrices and load vectors and assemble the stiffness matrix, we get

$$K = \begin{bmatrix} 0.875 * 10^5 & -0.875 * 10^5 & 0 \\ -0.875 * 10^5 & (0.875 * 10^5 + 0.625 * 10^5) & -0.625 * 10^5 \\ 0 & -0.625 * 10^5 & 0.625 * 10^5 \end{bmatrix}$$

Global load vector due to body force is

$$F_{B.F} = \begin{Bmatrix} 0.241 \\ 0.241 + 0.172 \\ 0.172 \end{Bmatrix} = \begin{Bmatrix} 0.241 \\ 0.413 \\ 0.172 \end{Bmatrix} \text{ N}$$

We have global load vector due to point load as

$$P = \begin{Bmatrix} 0 \\ +100 \\ 0 \end{Bmatrix} \text{ N}$$

Then, total global load vector due to body force and point load is

$$F = F_{B.F} + P = \begin{Bmatrix} 0.241 \\ 100.413 \\ 0.172 \end{Bmatrix} \text{ N}$$

Global system equation can be written as

$$\begin{bmatrix} -0.875 \times 10^5 & -0.875 \times 10^5 & 0 \\ -0.875 \times 10^5 & 1.5 \times 10^5 & -0.625 \times 10^5 \\ 0 & -0.625 \times 10^5 & 0.625 \times 10^5 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 0.241 \\ 100.413 \\ 0.172 \end{Bmatrix} \quad (A)$$

d) To solve the system equation:

Application of boundary condition: As the plate is at node 1, we have $U_1 = a_1 = 0$. Therefore, using elimination method Eq. (A) can be modified by eliminating 1st row and 1st column. The modified system equation is written as

$$\begin{bmatrix} 1.5 \times 10^5 & -0.625 \times 10^5 \\ -0.625 \times 10^5 & 0.625 \times 10^5 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 100.413 - K_{21}a_1 \\ 0.172 - K_{31}a_1 \end{Bmatrix} = \begin{Bmatrix} 100.413 \\ 0.172 \end{Bmatrix} \quad (\text{Since, } a_1 = 0)$$

The above equation can be written in expanded form as

$$1.5 \times 10^5 U_2 - 0.625 \times 10^5 U_3 = 100.413 \quad (i)$$

$$-0.625 \times 10^5 U_2 + 0.625 \times 10^5 U_3 = 0.172 \quad (ii)$$

Add (i) and (ii), we get $0.875 \times 10^5 U_2 = 100.585$

$\therefore U_2 = 1.14 \times 10^{-3} \text{ mm}$. Substitute U_2 into Eq. (i) or (ii) and solve for U_3 , we get

$$U_3 = 1.15 \times 10^{-3} \text{ mm}$$

Therefore, displacement vector is $Q \begin{Bmatrix} 0 & 1.14 \times 10^{-3} & 1.15 \times 10^{-3} \end{Bmatrix}^T \text{ mm}$

e) To evaluate element stresses: In general element stress is given by

$\sigma^e = E_e B^e Q^e$, where $B^e = \begin{bmatrix} -\frac{1}{l_e} & \frac{1}{l_e} \end{bmatrix}$ and Q^e is the nodal displacement vector corresponding to e^{th} element.

$$\begin{aligned} \text{For element 1: } \sigma^1 &= E_1 \begin{bmatrix} -\frac{1}{l_1} & \frac{1}{l_1} \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = 200 \times 10^3 \begin{bmatrix} -\frac{1}{120} & \frac{1}{120} \end{bmatrix} \begin{Bmatrix} 0 \\ 1.14 \times 10^{-3} \end{Bmatrix} \\ &= 200 \times 10^3 \times \frac{1}{120} \times 1.14 \times 10^{-3} = 1.9 \text{ N/mm}^2 \end{aligned}$$

$$\begin{aligned} \text{For element 2: } \sigma^2 &= E_2 \begin{bmatrix} -\frac{1}{l_2} & \frac{1}{l_2} \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = 200 \times 10^3 \begin{bmatrix} -\frac{1}{120} & \frac{1}{120} \end{bmatrix} \begin{Bmatrix} 1.14 \times 10^{-3} \\ 1.15 \times 10^{-3} \end{Bmatrix} \\ &= 0.017 \text{ N/mm}^2 \end{aligned}$$

f) To determine support reaction: In elimination method, the support reactions at node 1 and 3 are obtained as follows:

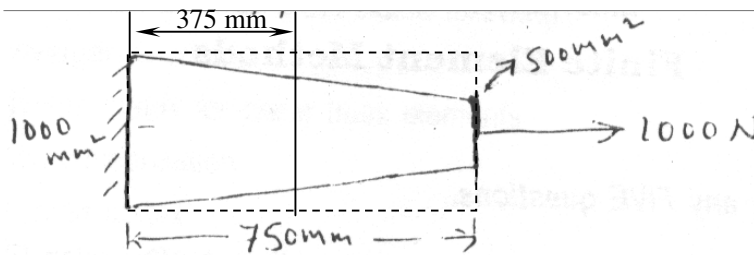
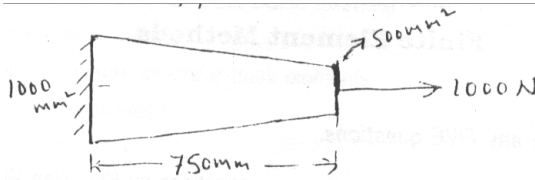
$$\text{At support 1: } K_{11}U_1 + K_{12}U_2 + K_{13}U_3 - F_1 = R_1$$

Or it is simply obtained by adding R_1 to the RHS of 1st equation of Eq. (A) and is written as

$$0.875 \times 10^5 U_1 - 0.875 \times 10^5 U_2 = 0.241 + R_1$$

$$\text{Or } R_1 = -0.875 \times 10^5 \times 1.14 \times 10^{-3} - 0.241 = -99.99 \approx -100 \text{ N} \quad \uparrow$$

Prob.2: For the following figure (bar), find the nodal displacements. The cross sectional area decreases linearly from 1000 mm^2 to 500 mm^2 . Use two elements. Take $E = 2 \times 10^6 \text{ Mpa}$, (VTU Jan. 2003)

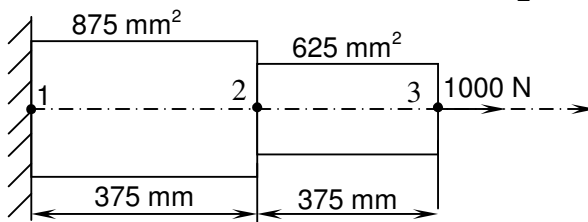


$$\text{Solution: } \tan \theta = \frac{a}{375} = \frac{250}{750} \quad \therefore a = \frac{250 \times 375}{750} = 125 \text{ mm}$$

$$\text{C/s area at midpoint} = 1000 - 2 \times a = 1000 - 2 \times 125 = 750 \text{ mm}^2$$

$$\text{Average C/s area of element 1, } A_1 = \frac{1000 + 750}{2} = 875 \text{ mm}^2$$

$$\text{Average C/s area of element 2, } A_2 = \frac{750 + 500}{2} = 625 \text{ mm}^2$$



Proceed similar to the previous problem and determine nodal displacement.

Prob 3: A load $P = 60 \text{ kN}$ is applied as shown in Fig (a). Determine the displacement field, stress and support reactions in the body. Take $E = 20 \times 10^3 \text{ N/mm}^2$. Use (a) Elimination method and (b) Penalty method to handle the boundary conditions.

Solution: Divide the given bar into two elements such that the point load lies on a node point as shown in Fig. (b).

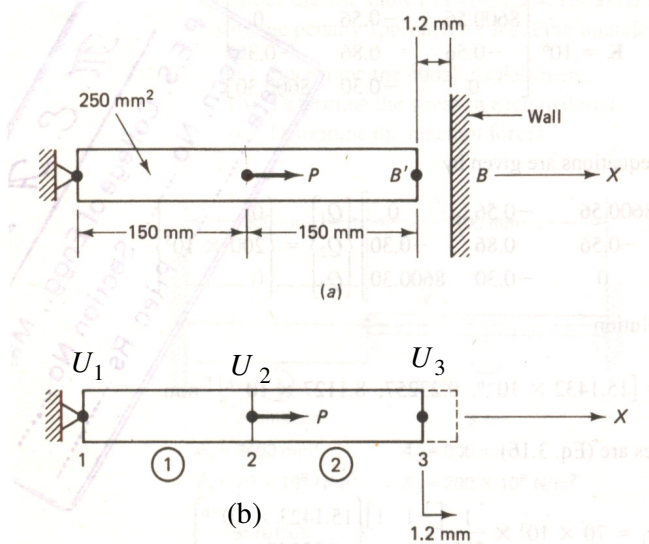
In general element stiffness matrix for an e^{th} element is expressed as

$$K^e = \frac{A_e E_e}{l_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Therefore,

For element 1:

$$K^1 = \frac{20 \times 10^3 \times 250}{150} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0.33 \times 10^5 & -0.33 \times 10^5 \\ -0.33 \times 10^5 & 0.33 \times 10^5 \end{bmatrix}$$



For element 2:

$$K^2 = K^1 = \begin{bmatrix} 0.33 \times 10^5 & -0.33 \times 10^5 \\ -0.33 \times 10^5 & 0.33 \times 10^5 \end{bmatrix} \quad \text{since, } A_1=A_2, \quad E_1=E_2 \text{ and } l_1=l_2$$

Global stiff. Matrix is

$$K = \begin{bmatrix} 0.33 \times 10^5 & -0.33 \times 10^5 & 0 \\ -0.33 \times 10^5 & (0.33 \times 10^5 + 0.33 \times 10^5) & -0.33 \times 10^5 \\ 0 & -0.33 \times 10^5 & 0.33 \times 10^5 \end{bmatrix} \text{ N/mm}$$

As there are no body and traction forces, the total global load vector is only due to point load and is given by

$$F = \begin{Bmatrix} 0 \\ 60 \times 10^3 \\ 0 \end{Bmatrix} \text{ N}$$

Global system equation is

$$\begin{bmatrix} 0.33 \times 10^5 & -0.33 \times 10^5 & 0 \\ -0.33 \times 10^5 & 0.66 \times 10^5 & -0.33 \times 10^5 \\ 0 & -0.33 \times 10^5 & 0.33 \times 10^5 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 60 \times 10^3 \\ 0 \end{Bmatrix} \quad (A)$$

Application of boundary conditions:

(a) Elimination method: Here $U_1 = a_1 = 0.0$ and $U_3 = a_3 = 1.2 \text{ mm}$

Hence, using elimination method, Eq (A) is modified by eliminating 1st row and 1st column and 3rd row and 3rd column and is given by

$$[0.66 \times 10^5] \{U_2\} = \{60 \times 10^3 - K_{21} \cdot a_1 - K_{23} \cdot a_3\} = \{60 \times 10^3 + 0.33 \times 10^5 \cdot 0 + 0.33 \times 10^5 \cdot 1.2\}$$

$$0.66 \times 10^5 U_2 = 0.996 \times 10^5$$

$$\therefore U_2 = 1.5 \text{ mm}$$

Therefore, nodal displacement field is $Q = \{U_1 \quad U_2 \quad U_3\}^T = \{0 \quad 1.5 \quad 1.2\}^T \text{ mm}$

Elemental stresses: In general, the elemental stress is expressed as

$$\sigma^e = E_e \varepsilon^e = E_e B^e Q^e$$

Then,

For element 1:

$$\sigma^1 = E_1 \begin{bmatrix} -\frac{1}{l_1} & \frac{1}{l_1} \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = 20 * 10^3 \begin{bmatrix} -\frac{1}{150} & \frac{1}{150} \end{bmatrix} \begin{Bmatrix} 0 \\ 1.5 \end{Bmatrix} = 20 * 10^3 * \frac{1}{150} * 1.5$$

= 200 N/mm² (tensile)

For element 2:

$$\sigma^2 = E_2 \begin{bmatrix} -\frac{1}{l_2} & \frac{1}{l_2} \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = 20 * 10^3 \begin{bmatrix} -\frac{1}{150} & \frac{1}{150} \end{bmatrix} \begin{Bmatrix} 1.5 \\ 1.2 \end{Bmatrix} = 20 * 10^3 \left(-\frac{1}{150} * 1.5 + \frac{1}{150} * 1.2 \right)$$

= - 40 N/mm² (compressive)

Support reactions: In elimination method, the support reactions at node 1 and 3 are obtained as follows:

$$\text{At support 1: } K_{11}U_1 + K_{12}U_2 + K_{13}U_3 - F_1 = R_1$$

Or it is simply obtained by adding R_1 to the RHS of 1st equation of Eq. (A) and is written as

$$0.33 * 10^5 U_1 - 0.33 * 10^5 U_2 = 0 + R_1$$

$$\text{Or } R_1 = -0.33 * 10^5 * 1.5 = -49500 \text{ N (} \longleftarrow \text{)}$$

$$\text{At support 2: } K_{31}U_1 + K_{32}U_2 + K_{33}U_3 - F_3 = R_3$$

Or it is obtained by adding R_3 to the RHS of 3rd equation of Eq. (A)

$$-0.33 * 10^5 * U_2 + 0.33 * 10^5 * U_3 = 0 + R_3$$

$$\text{Or } R_3 = (-0.33 * 1.5 + 0.33 * 1.2) 10^5 = -9900 \text{ N (} \longleftarrow \text{)}$$

(b) Penalty approach: Here $U_1 = a_1 = 0.0$ and $U_3 = a_3 = 1.2 \text{ mm}$

$$C = \text{Max} |K_{ij}| 10^4$$

$$\text{From Eq. (A), } \text{Max} |K_{ij}| = 0.66 * 10^5 \therefore C = 0.66 * 10^5 * 10^4 = 0.66 * 10^9$$

Hence, using penalty approach, Eq. (A) is modified as

$$\begin{bmatrix} 0.33 \times 10^5 + 0.66 * 10^9 & -0.33 \times 10^5 & 0 \\ -0.33 \times 10^5 & 0.66 \times 10^5 & -0.33 \times 10^5 \\ 0 & -0.33 \times 10^5 & 0.33 \times 10^5 + 0.66 * 10^9 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 0 + 0.66 * 10^9 * 0 \\ 60 * 10^3 \\ 0 + 0.66 * 10^9 * 1.2 \end{Bmatrix}$$

Solve the above equation for unknown U s, we get

$$U_1 = 7.5 * 10^{-5} \text{ mm}; U_2 = 1.500045 \text{ mm} \text{ and } U_3 = 1.200015 \text{ mm}$$

Elemental stresses: In general, the elemental stress is expressed as

$$\sigma^e = E_e \varepsilon^e = E_e B^e Q^e$$

Then,

For element 1:

$$\sigma^1 = E_1 \begin{bmatrix} -\frac{1}{l_1} & \frac{1}{l_1} \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = 20 * 10^3 \begin{bmatrix} -\frac{1}{150} & \frac{1}{150} \end{bmatrix} \begin{Bmatrix} 7.5 * 10^{-5} \\ 1.500045 \end{Bmatrix}$$

$$= 20 * 10^3 \left(-\frac{1}{150} * 7.5 * 10^{-5} + \frac{1}{150} * 1.500045 \right) = 200.016 \text{ N/mm}^2 \text{ (tensile)}$$

For element 2:

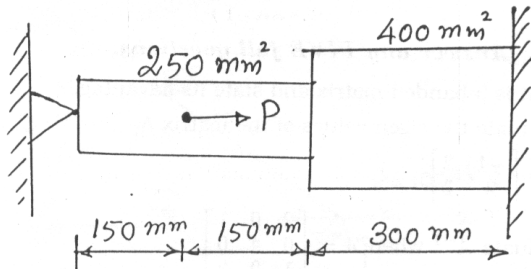
$$\begin{aligned}\sigma^2 &= E_2 \begin{bmatrix} -\frac{1}{l_2} & \frac{1}{l_2} \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = 20 \times 10^3 \begin{bmatrix} -\frac{1}{150} & \frac{1}{150} \end{bmatrix} \begin{Bmatrix} 1.500045 \\ 1.200015 \end{Bmatrix} \\ &= 20 \times 10^3 \left(-\frac{1}{150} \times 1.500045 + \frac{1}{150} \times 1.200015 \right) = -40.004 \text{ N/mm}^2 \text{ (compressive)}\end{aligned}$$

Support reactions: In penalty approach, the support reactions at node 1 and 3 are obtained as follows:

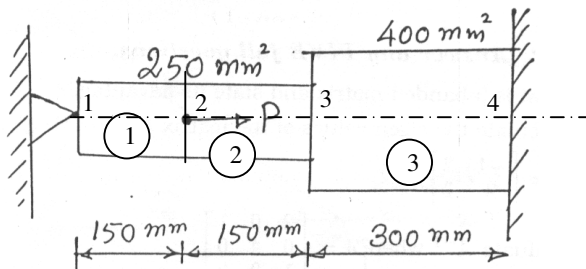
$$\text{At support 1: } R_1 = -C(U_1 - a_1) = -0.66 \times 10^9 (7.5 \times 10^{-5} - 0) = -49500 \text{ N}$$

$$\text{At support 2: } R_2 = -C(U_3 - a_3) = -0.66 \times 10^9 (1.200015 - 1.2) = -9900 \text{ N}$$

Prob. 4: Determine the nodal displacements, element stresses and support reactions of the axially loaded bar shown in Fig. Use elimination method for handling the boundary conditions. Take $E = 200 \text{ Gpa}$ and load $P = 300 \text{ kN}$. (VTU Feb. 2002)



Solution: Divide the given system into two elements so such that point load lies on a node point as shown below.



$$\begin{aligned}E &= 200 \times 10^3 \text{ N/mm}^2 \\ P &= 300 \times 10^3 \text{ N}\end{aligned}$$

Elemental stiffness matrices:

$$K^e = \frac{A_e E_e}{l_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

For element 1:

$$K^1 = \frac{200 \times 10^3 \times 250}{150} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3.33 \times 10^5 & -3.33 \times 10^5 \\ -3.33 \times 10^5 & 3.33 \times 10^5 \end{bmatrix}$$

For element 2:

$$K^2 = K^1 = \begin{bmatrix} 3.33 \times 10^5 & -3.33 \times 10^5 \\ -3.33 \times 10^5 & 3.33 \times 10^5 \end{bmatrix} \quad \text{since, } A_1 = A_2, E_1 = E_2 \text{ and } l_1 = l_2$$

For element 3:

$$K^3 = \frac{200 \times 10^3 \times 400}{300} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2.67 \times 10^5 & -2.67 \times 10^5 \\ -2.67 \times 10^5 & 2.67 \times 10^5 \end{bmatrix}$$

Global stiff. Matrix is

$$K = \begin{bmatrix} 3.33 \times 10^5 & -3.33 \times 10^5 & 0 & 0 \\ -3.33 \times 10^5 & (3.33 \times 10^5 + 3.33 \times 10^5) & -3.33 \times 10^5 & 0 \\ 0 & -3.33 \times 10^5 & (3.33 \times 10^5 + 2.67 \times 10^5) & -2.67 \times 10^5 \\ 0 & 0 & -2.67 \times 10^5 & 2.67 \times 10^5 \end{bmatrix} \text{ N/mm}$$

As there are no body and traction forces, the total global load vector is only due to point load and is given by

$$F = \begin{Bmatrix} 0 \\ 300 \times 10^3 \\ 0 \\ 0 \end{Bmatrix} \text{ N}$$

Global system equation is

$$\begin{bmatrix} 3.33 \times 10^5 & -3.33 \times 10^5 & 0 & 0 \\ -3.33 \times 10^5 & 6.66 \times 10^5 & -3.33 \times 10^5 & 0 \\ 0 & -3.33 \times 10^5 & 6 \times 10^5 & -2.67 \times 10^5 \\ 0 & 0 & -2.67 \times 10^5 & 2.67 \times 10^5 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 300 \times 10^3 \\ 0 \\ 0 \end{Bmatrix} \quad (A)$$

Application of boundary conditions:

(a) Elimination method: Here $U_1 = a_1 = 0.0$ and $U_4 = a_4 = 0$

Hence, using elimination method, Eq (A) is modified by eliminating 1st row and 1st column and 4th row and 4th column and is given by

$$\begin{bmatrix} 6.66 \times 10^5 & -3.33 \times 10^5 \\ -3.33 \times 10^5 & 6 \times 10^5 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 300 \times 10^3 - K_{21}a_1 - K_{24}a_4 \\ 0 - K_{31}a_1 - K_{34}a_4 \end{Bmatrix} = \begin{Bmatrix} 300 \times 10^3 \\ 0 \end{Bmatrix} \quad \text{since,}$$

$$a_1 = a_4 = 0$$

$$6.66 \times 10^5 U_2 - 3.33 \times 10^5 U_3 = 300 \times 10^3 \quad (i)$$

$$-3.33 \times 10^5 U_2 + 6 \times 10^5 U_3 = 0$$

Or $U_2 = 1.802 U_3$. Substitute into Eq. (i), we get

$$6.66 \times 10^5 \times 1.802 U_3 - 3.33 \times 10^5 U_3 = 300 \times 10^3$$

$$\therefore U_3 = 0.346 \text{ mm and } U_2 = 1.802 \times 0.346 = 0.624 \text{ mm}$$

Thus, nodal displacements are $Q = \{0 \quad 0.624 \quad 0.346 \quad 0\}^T \text{ mm}$

Element Stress: $\sigma^e = E_e B^e Q^e$

$$\sigma^1 = E_1 \begin{bmatrix} -\frac{1}{l_1} & \frac{1}{l_1} \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = 200 \times 10^3 \begin{bmatrix} -\frac{1}{150} & \frac{1}{150} \end{bmatrix} \begin{Bmatrix} 0 \\ 0.624 \end{Bmatrix} = 832 \text{ N/mm}^2 \text{ (tensile)}$$

$$\sigma^2 = E_2 \begin{bmatrix} -\frac{1}{l_2} & \frac{1}{l_2} \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = 200 \times 10^3 \begin{bmatrix} -\frac{1}{150} & \frac{1}{150} \end{bmatrix} \begin{Bmatrix} 0.624 \\ 0.346 \end{Bmatrix} = -370.67 \text{ N/mm}^2 \text{ (compressive)}$$

$$\sigma^3 = E_3 \begin{bmatrix} -\frac{1}{l_3} & \frac{1}{l_3} \end{bmatrix} \begin{Bmatrix} U_3 \\ U_4 \end{Bmatrix} = 200 \times 10^3 \begin{bmatrix} -\frac{1}{300} & \frac{1}{300} \end{bmatrix} \begin{Bmatrix} 0.346 \\ 0 \end{Bmatrix} = -230.67 \text{ N/mm}^2 \text{ (compressive)}$$

Support Reactions:

At support 1: Add R_1 to the RHS of 1st equation of Eq. (A), we get

$$R_1 = 3.33 \times 10^5 U_1 - 3.33 \times 10^5 U_2 = -3.33 \times 10^5 \times 0.624 = -207792 \text{ N} \quad (\text{here } U_1=0)$$

At support 4: Add R_4 to the RHS of 4th equation of Eq. (A), we get

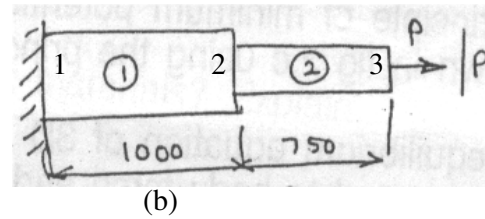
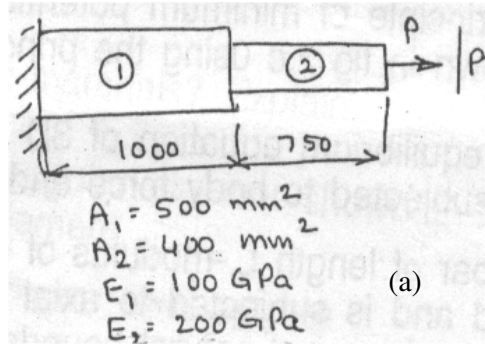
$$R_4 = -2.67 * 10^5 U_3 + 2.67 * 10^5 U_4 = -2.67 * 10^5 * 0.346 = -92382 \text{ N} \quad (\text{here } U_4=0)$$

Prob. 5: A component shown in Fig. is subjected to a load of 5 kN. Determine the following.

i) Element stiffness matrices ii) B – Matrices iii) Displacements and strains

iv) Stresses and reactions.

Obtain the stiffness matrix and load vector assuming two elements. (VTU July 2004)



i) **Element stiffness matrices:**

For element 1:

$$K^1 = \frac{500 * 100 * 10^3}{1000} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 50 * 10^3 & -50 * 10^3 \\ -50 * 10^3 & 50 * 10^3 \end{bmatrix}$$

For element 2:

$$K^2 = \frac{400 * 200 * 10^3}{750} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 106.67 * 10^3 & -106.67 * 10^3 \\ -106.67 * 10^3 & 106.67 * 10^3 \end{bmatrix}$$

ii) **B – Matrices:** In general, the B – Matrix for 2-node bar element is expressed as

$$B^e = \begin{bmatrix} -\frac{1}{l_e} & \frac{1}{l_e} \end{bmatrix}$$

For element 1, $B^1 = \begin{bmatrix} -\frac{1}{1000} & \frac{1}{1000} \end{bmatrix}$ and for element 2, $B^2 = \begin{bmatrix} -\frac{1}{750} & \frac{1}{750} \end{bmatrix}$

iii) **Displacements and strains:** The displacements and strains are obtained by the assembly process and solution of assembled system equation. The assembled or global stiffness matrix is obtained as

$$K = \begin{bmatrix} 50 * 10^3 & -50 * 10^3 & 0 \\ -50 * 10^3 & (50 * 10^3 + 106.67 * 10^3) & -106.67 * 10^3 \\ 0 & -106.67 * 10^3 & 106.67 * 10^3 \end{bmatrix}$$

Load vector due to point load only is $F = \begin{Bmatrix} 0 & 0 & 5 * 10^3 \end{Bmatrix}^T \text{ N}$

Therefore, the system equation becomes as

$$\begin{bmatrix} 50 * 10^3 & -50 * 10^3 & 0 \\ -50 * 10^3 & 156.67 * 10^3 & -106.67 * 10^3 \\ 0 & -106.67 * 10^3 & 106.67 * 10^3 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 5 * 10^3 \end{Bmatrix} \quad (A)$$

Application of boundary conditions: Boundary condition is $U_1 = 0$. Therefore, using elimination method Eq. (A) can be modified by eliminating 1st row and 1st column and is written as

$$\begin{bmatrix} 156.67 \times 10^3 & -106.67 \times 10^3 \\ -106.67 \times 10^3 & 106.67 \times 10^3 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 5 \times 10^3 \end{Bmatrix}$$

$$156.67 \times 10^3 U_2 - 106.67 \times 10^3 U_3 = 0 \quad (i)$$

$$-106.67 \times 10^3 U_2 + 106.67 \times 10^3 U_3 = 5 \times 10^3 \quad (ii)$$

Add (i) and (ii), we get

$$50 \times 10^3 U_2 = 5 \times 10^3. \text{ Therefore, } U_2 = 0.1 \text{ mm}$$

Substitute U_2 into Eq. (i) and solve, we get $U_3 = 0.147 \text{ mm}$.

Therefore, displacements field is $Q = \{0 \quad 0.1 \quad 0.147\}^T \text{ mm}$

To determine strains: we have $\epsilon^e = B^e Q^e$

$$\therefore \epsilon^1 = \begin{bmatrix} -\frac{1}{1000} & \frac{1}{1000} \end{bmatrix} \begin{Bmatrix} 0 \\ 0.1 \end{Bmatrix} = 1 \times 10^{-4} \text{ and } \therefore \epsilon^2 = \begin{bmatrix} -\frac{1}{750} & \frac{1}{750} \end{bmatrix} \begin{Bmatrix} 0.1 \\ 0.147 \end{Bmatrix} = 0.627 \times 10^{-4}$$

iv) Stresses and reactions:

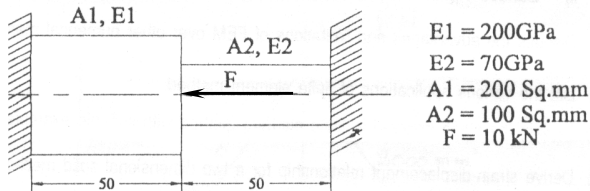
Stresses are $\sigma^1 = E_1 \epsilon^1 = 100 \times 10^3 \times 1 \times 10^{-4} = 10 \text{ N/mm}^2$ and

$$\sigma^2 = E_2 \epsilon^2 = 200 \times 10^3 \times 0.627 \times 10^{-4} = 12.53 \text{ N/mm}^2$$

Reaction is obtained by adding R_1 to the RHS of 1st equation of Eq. (A) as

$$R_1 = -50 \times 10^3 \times U_2 = -50 \times 10^3 \times 0.1 = -5 \text{ kN}$$

Prob. 6: Obtain the displacement field, stresses and support reactions for the stepped bar shown in Fig. Use elimination and penalty approach to handle the boundary conditions. Verify the answer.



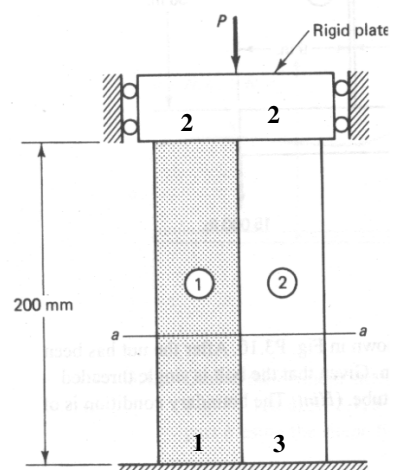
Hint: Obtain the global system equation similar to the previous problems and solve separately using elimination method and penalty approach. The global load vector in this case is

$F = \{0 \quad -10 \times 10^3 \quad 0\}^T \text{ N}$, negative sign for the applied load is because it acts in negative x-direction.

Prob. 7: Consider the composite structure of axially loaded member shown in Fig. Determine unknown displacements, stresses and reaction forces.

$$E_1 = 70 \text{ GPa}, E_2 = 105 \text{ GPa}, A_1 = A_2 = 1800 \text{ mm}^2, P = 385 \text{ kN}.$$

Solution: Since all bars in composite structure have same displacement (deformation), the given structure can be discretized using single node at the top of the bars as shown. It may be noted that, although the displacement of two bars at the support are same two nodes are used to determine support reactions which are different under different bars.



$$K^1 = \frac{1800 \times 70 \times 10^3}{200} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 6.3 \times 10^5 & -6.3 \times 10^5 \\ -6.3 \times 10^5 & 6.3 \times 10^5 \end{bmatrix}$$

$$K^2 = \frac{1800 * 105 * 10^3}{200} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 9.45 * 10^5 & -9.45 * 10^5 \\ -9.45 * 10^5 & 9.45 * 10^5 \end{bmatrix}$$

Connectivity table

Element Nos.	Local node Nos.	Global D.O.F
1	1	1
	2	2
2	1	3
	2	2

$$\begin{bmatrix} 6.3 * 10^5 & -6.3 * 10^5 & 0 \\ -6.3 * 10^5 & 15.75 * 10^5 & -9.45 * 10^5 \\ 0 & -9.45 * 10^5 & 9.45 * 10^5 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -385 * 10^3 \\ 0 \end{Bmatrix}$$

Boundary conditions: $U_1 = 0$ and $U_3 = 0$. Eliminate 1st row and 1st column and 3rd row and 3rd column, we get

$$15.75 * 10^5 U_2 = -385 * 10^3$$

$$\therefore U_2 = -0.244 \text{ mm (Node 2 moves towards -ve x-direction)}$$

Stresses:

$$\sigma^1 = E_1 \begin{bmatrix} -\frac{1}{l_1} & \frac{1}{l_2} \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = 70 * 10^3 \begin{bmatrix} -\frac{1}{200} & \frac{1}{200} \end{bmatrix} \begin{Bmatrix} 0 \\ -0.244 \end{Bmatrix} = -85.4 \text{ N/mm}^2$$

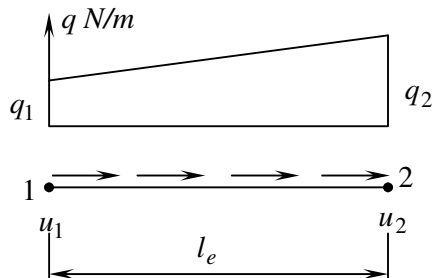
$$\begin{aligned} \sigma^2 &= E_2 \begin{bmatrix} -\frac{1}{l_2} & \frac{1}{l_2} \end{bmatrix} \begin{Bmatrix} U_3 \\ U_2 \end{Bmatrix}^{***} = 105 * 10^3 \begin{bmatrix} -\frac{1}{200} & \frac{1}{200} \end{bmatrix} \begin{Bmatrix} 0 \\ -0.244 \end{Bmatrix} \\ &= \frac{-105 * 10^3 * 0.244}{200} = -256.2 \text{ N/mm}^2 \end{aligned}$$

*** Note that the nodal displacements corresponding to the local node 1 and 2 of element 2 are U_3 and U_2 respectively.

Support reactions:

$$R_1 = -6.3 * 10^5 * -0.244 = +153.72 \text{ kN} \uparrow \text{ and } R_3 = -9.45 * 10^5 * -0.244 = 230.58 \text{ kN} \uparrow$$

Prob. 8: A bar element carries a distributed load of q N/m, which varies from q_1 at one end to q_2 at other end. Calculate consistent nodal loads. (VTU May 2007)



Consider a 2 – noded bar element subjected to an uniformly varying traction force which varies from q_1 to q_2 at node 1 and 2 respectively as shown in Fig. Then, load vector due to this traction force can be obtained from work potential due to surface force.

$$W_{S.F} = \int_s u^T T ds = \int_0^{l_e} u^T q dx \quad (\text{for bar element } T = T_x^e = q \text{ and } ds = dx)$$

$$u = N_1 u_1 + N_2 u_2 = [N_1 \quad N_2] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = N q_u$$

$$\therefore u^T = q_u^T N^T$$

Again, since traction force varies from one node another node, it can be interpolated as

$$q = N_1 q_1 + N_2 q_2 = [N_1 \quad N_2] \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

Substitute u^T and q into $W_{S.F}$, we get

$$W_{S.F} = q_u^T \int_0^{l_e} N^T [N_1 \quad N_2] \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} dx = q_u^T f^e$$

where, f^e is consistent nodal load vector and is expressed as

$$f^e = \int_0^{l_e} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} [N_1 \quad N_2] \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} dx = \int_0^{l_e} \begin{bmatrix} N_1^2 & N_1 N_2 \\ N_1 N_2 & N_2^2 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} dx$$

Using the integration rule for bar element, it can be shown that

$$\int_0^{l_e} N_1^2 dx = \int_0^{l_e} N_2^2 dx = \frac{2!}{(2+1)!} l_e = \frac{1*2}{1*2*3} l_e = \frac{1}{3} l_e \quad \text{and} \quad \int_0^{l_e} N_1 N_2 dx = \frac{1!!}{(1+1+1)!} l_e = \frac{1}{6} l_e$$

$$\text{Substitute into above equation we get } f^e = \begin{bmatrix} \frac{l_e}{3} & \frac{l_e}{6} \\ \frac{l_e}{6} & \frac{l_e}{3} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \frac{l_e}{6} \begin{Bmatrix} 2q_1 + q_2 \\ q_1 + 2q_2 \end{Bmatrix}$$

16. Isoparametric Elements:

For the analysis of elasticity problems which are having complex or curved shapes, simple triangular or rectangular elements are no longer sufficient. This needs to develop an element of arbitrary shape called *Isoparametric element*.

The concept of isoparametric element is based on the transformation of the parent element in local or natural coordinate system to an arbitrary shape in cartesian coordinate system as shown in Fig. 10. This can be done by using shape functions and nodal values of cartesian coordinates. Thus the cartesian coordinate of any point in an element may be expressed as

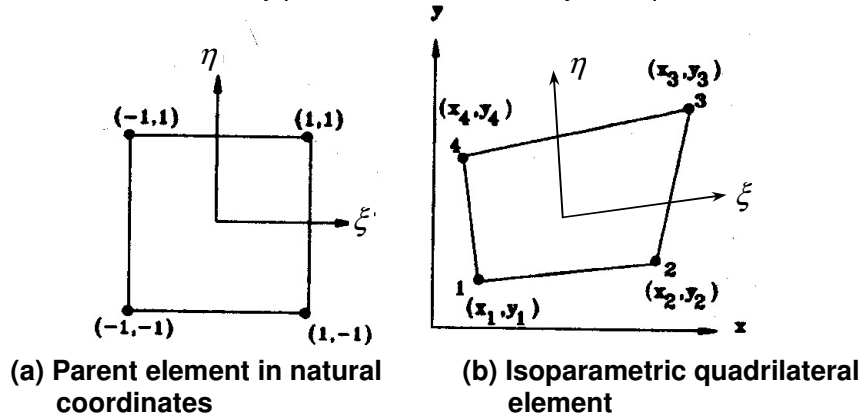


Fig. 10: 4-noded Isoparametric quadrilateral element

$$x = N'_1 x_1 + N'_2 x_2 + N'_3 x_3 + N'_4 x_4$$

$$x = [N'] \{x_m\} \text{ and similarly } y = [N'] \{y_m\} \quad \text{---(a)}$$

Further, the variation of displacement at any point within an element can also be expressed as

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3 + N_4 u_4$$

$$u = [N] \{q_u\} \text{ and similarly } v = [N] \{q_v\} \quad \text{---(b)}$$

If the shape functions $[N']$ in Eqn. (a) are same as the shape functions $[N]$ in Eqn. (b), then the element is called as *Isoparametric element*.

Thus, the isoparametric element is "one in which same functions used to define the geometry of the element and variation of displacements within the elements are same".

Super-parametric element: Is one which is having higher order shape functions used to define geometry than the shape functions used to define the variation of displacement. In case of superparametric element the number of nodes used to define the geometry of the element are more than the number of nodes used to define the variation of displacement as shown in Fig. 11.

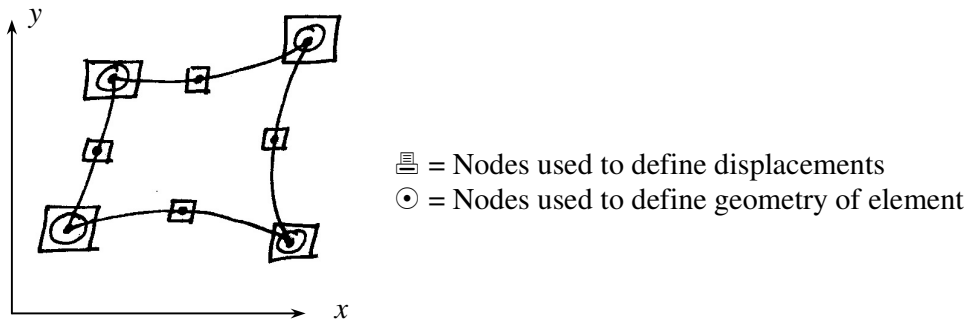


Fig. 11: Super-parametric element

Let, N' = Shape functions used to define variation of geometry of element.

N = Shape functions used to define variation of displacement.

Then, $N' > N$

Sub-parametric element: Is one, which having lower order shape functions to define geometry of the element than the shape functions used to define variation of displacement. In case of sub-parametric element the number of nodes used to define the geometry of the element are less than the number of nodes used to define the variation of displacement as shown in Fig. 12.

Let, N' = Shape functions used to define variation of geometry of element.

N = Shape functions used to define variation of displacement.

Then, $N' < N$

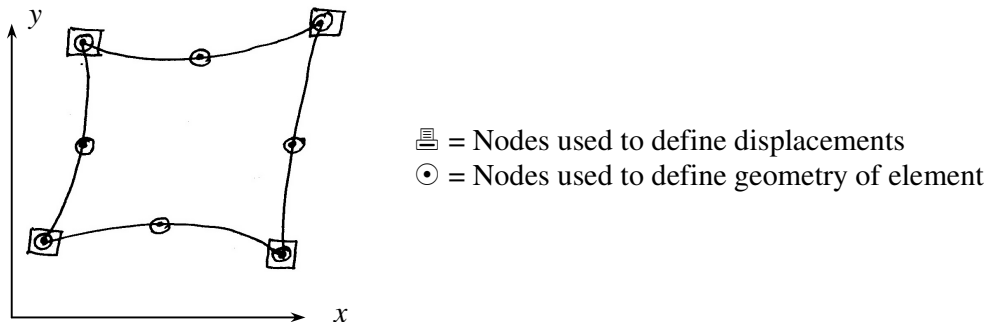


Fig. 12: Sub-parametric element

15. Jacobian Matrix:

While evaluating the strains, partial derivatives of u and v are to be taken w.r.t x and y . The

shape functions used to define the variation of displacement and geometry of the element are expressed in natural coordinates ξ and η for the case of 2-D elements. Hence, u, v and x, y are the functions of ξ and η . Therefore, the relationship between the two coordinate systems can be computed by using chain rule of partial derivatives as given below.

$$\begin{aligned}\frac{\partial u}{\partial \xi} &= \frac{\partial u}{\partial x} \times \frac{\partial x}{\partial \xi} + \frac{\partial u}{\partial y} \times \frac{\partial y}{\partial \xi} \text{ and} \\ \frac{\partial u}{\partial \eta} &= \frac{\partial u}{\partial x} \times \frac{\partial x}{\partial \eta} + \frac{\partial u}{\partial y} \times \frac{\partial y}{\partial \eta} \\ \begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \end{Bmatrix} &= \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{Bmatrix} \\ \begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \end{Bmatrix} &= [J] \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{Bmatrix} \quad \text{---(a)}\end{aligned}$$

where, matrix J is called *Jacobian matrix* or *Jacobian of transformation*.

Thus, Jacobian matrix is a *transformation matrix* used to transform natural coordinate to cartesian coordinate.

Note:

1. Using this matrix the derivatives of u w.r.t x and y can be obtained (from Eqn. (a) as

$$\begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \end{Bmatrix}$$

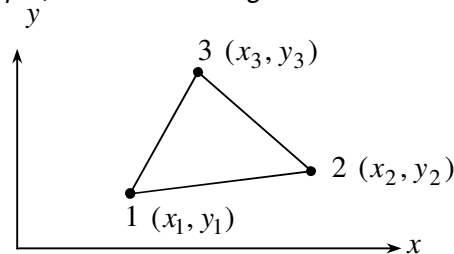
2. Similar to the above equation, we can also write

$$\begin{Bmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \frac{\partial v}{\partial \xi} \\ \frac{\partial v}{\partial \eta} \end{Bmatrix}$$

3. In general the Jacobian matrix for all 2-D linear element is given by

$$[J] = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}$$

4. The Jacobian matrix J at any point $P(x, y)$ inside the element can be expressed in terms of nodal coordinates. For example, consider a triangular element shown in Fig.



The Jacobian matrix for a 2-D element is $[J] = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}$ ---(a)

Also from Isoparametric formulation we can write

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3 = \xi x_1 + \eta x_2 + (1 - \xi - \eta) x_3 \quad \text{---(b)}$$

where, $N_1 = \xi$; $N_2 = \eta$ and $N_3 = (1 - \xi - \eta)$ for a triangular element.

$$\text{Similarly, } y = N_1 y_1 + N_2 y_2 + N_3 y_3 = \xi y_1 + \eta y_2 + (1 - \xi - \eta) y_3 \quad \text{---(c)}$$

Differentiate Eqn. (b) w.r.t ξ and η and use the notation $x_{ij} = x_i - x_j$

$$\frac{\partial x}{\partial \xi} = x_1 - x_3 = x_{13} \quad \text{and} \quad \frac{\partial x}{\partial \eta} = x_2 - x_3 = x_{23}$$

Differentiate Eqn. (c) w.r.t ξ and η and use the notation $y_{ij} = y_i - y_j$

$$\frac{\partial y}{\partial \xi} = y_1 - y_3 = y_{13} \quad \text{and} \quad \frac{\partial y}{\partial \eta} = y_2 - y_3 = y_{23}$$

Substitute into Jacobian matrix Eqn. (a), we get

$$[J] = \begin{bmatrix} x_{13} & y_{13} \\ x_{23} & y_{23} \end{bmatrix}$$

5. Area of the triangle can be obtained from Jacobian matrix as

$$A = \frac{1}{2} \det J = \frac{1}{2} (x_{13} * y_{23} - x_{23} * y_{13})$$

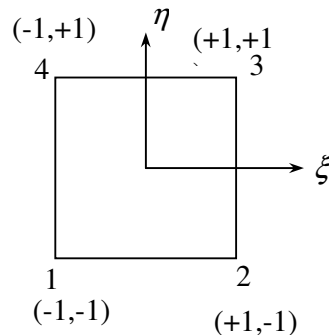
Problems:

Prob.1: Derive strain-displacement relation of a CST element. (VTU, Feb.2002)

Solution: Derive $\{\epsilon\} = [B]\{q\}$ for a triangular element.

Prob.2: For a linear quadrilateral element, derive an expression for Jacobian matrix. (VTU, Feb.2002)

Solution:



Consider linear quadrilateral element as in Fig. The field variable at any point within the element can be expressed as

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3 + N_4 u_4$$

$$v = N_1 v_1 + N_2 v_2 + N_3 v_3 + N_4 v_4 \quad \text{---(a)}$$

Similarly, the coordinates x- and y- of a point can be expressed as

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4$$

$$y = N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4 \quad \text{---(b)}$$

where, N_1, N_2, N_3 and N_4 are shape functions and are expressed as

$$\begin{aligned} N_1 &= \frac{1}{4}(1-\xi)(1-\eta) \\ N_2 &= \frac{1}{4}(1+\xi)(1-\eta) \\ N_3 &= \frac{1}{4}(1+\xi)(1+\eta) \\ N_4 &= \frac{1}{4}(1-\xi)(1+\eta) \end{aligned} \quad \text{---(c)}$$

Since, shape functions are functions of natural coordinates ξ and η the displacements u and v and coordinates x and y are also functions of these natural coordinates. Therefore, the relationship between the two coordinate systems can be computed by using chain rule of partial derivatives as given below.

$$\begin{aligned} \frac{\partial u}{\partial \xi} &= \frac{\partial u}{\partial x} \times \frac{\partial x}{\partial \xi} + \frac{\partial u}{\partial y} \times \frac{\partial y}{\partial \xi} \quad \text{and} \\ \frac{\partial u}{\partial \eta} &= \frac{\partial u}{\partial x} \times \frac{\partial x}{\partial \eta} + \frac{\partial u}{\partial y} \times \frac{\partial y}{\partial \eta} \\ \begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \end{Bmatrix} &= \begin{Bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{Bmatrix} \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{Bmatrix} \\ \begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \end{Bmatrix} &= [J] \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{Bmatrix} \end{aligned} \quad \text{---(d)}$$

where, matrix J is called *Jacobian matrix* and is given by

$$[J] = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \quad \text{---(e)}$$

Differentiate Eqn.(b) w.r.t ξ and η , we get

$$\begin{aligned} \frac{\partial x}{\partial \xi} &= \frac{\partial N_1}{\partial \xi} x_1 + \frac{\partial N_2}{\partial \xi} x_2 + \frac{\partial N_3}{\partial \xi} x_3 + \frac{\partial N_4}{\partial \xi} x_4 \\ \frac{\partial y}{\partial \xi} &= \frac{\partial N_1}{\partial \xi} y_1 + \frac{\partial N_2}{\partial \xi} y_2 + \frac{\partial N_3}{\partial \xi} y_3 + \frac{\partial N_4}{\partial \xi} y_4 \\ \frac{\partial x}{\partial \eta} &= \frac{\partial N_1}{\partial \eta} x_1 + \frac{\partial N_2}{\partial \eta} x_2 + \frac{\partial N_3}{\partial \eta} x_3 + \frac{\partial N_4}{\partial \eta} x_4 \\ \frac{\partial y}{\partial \eta} &= \frac{\partial N_1}{\partial \eta} y_1 + \frac{\partial N_2}{\partial \eta} y_2 + \frac{\partial N_3}{\partial \eta} y_3 + \frac{\partial N_4}{\partial \eta} y_4 \end{aligned}$$

Now, diff. Eqn.(c) w.r.t ξ and η , we get

$$\begin{aligned} \frac{\partial N_1}{\partial \xi} &= \frac{-(1-\eta)}{4}; \quad \frac{\partial N_2}{\partial \xi} = \frac{+(1-\eta)}{4}; \quad \frac{\partial N_3}{\partial \xi} = \frac{+(1+\eta)}{4}; \quad \frac{\partial N_4}{\partial \xi} = \frac{-(1+\eta)}{4} \\ \frac{\partial N_1}{\partial \eta} &= \frac{-(1-\xi)}{4}; \quad \frac{\partial N_2}{\partial \eta} = \frac{-(1+\xi)}{4}; \quad \frac{\partial N_3}{\partial \eta} = \frac{+(1+\xi)}{4}; \quad \frac{\partial N_4}{\partial \eta} = \frac{(1-\xi)}{4} \end{aligned}$$

Substitute into the above equations, we get

$$\begin{aligned}\frac{\partial x}{\partial \xi} &= -\frac{(1-\eta)}{4}x_1 + \frac{(1-\eta)}{4}x_2 + \frac{(1+\eta)}{4}x_3 - \frac{(1+\eta)}{4}x_4 \\ \frac{\partial y}{\partial \xi} &= -\frac{(1-\eta)}{4}y_1 + \frac{(1-\eta)}{4}y_2 + \frac{(1+\eta)}{4}y_3 - \frac{(1+\eta)}{4}y_4 \\ \frac{\partial x}{\partial \eta} &= -\frac{(1-\xi)}{4}x_1 - \frac{(1+\xi)}{4}x_2 + \frac{(1+\xi)}{4}x_3 + \frac{(1-\xi)}{4}x_4 \\ \frac{\partial y}{\partial \eta} &= -\frac{(1-\xi)}{4}y_1 - \frac{(1+\xi)}{4}y_2 + \frac{(1+\xi)}{4}y_3 + \frac{(1-\xi)}{4}y_4\end{aligned}$$

Substitution into Eqn.(e) yields Jacobian matrix as

$$[J] = \frac{1}{4} \begin{bmatrix} -(1-\eta)x_1 + (1-\eta)x_2 + (1+\eta)x_3 - (1+\eta)x_4 & -(1-\eta)y_1 + (1-\eta)y_2 + (1+\eta)y_3 - (1+\eta)y_4 \\ -(1-\xi)x_1 - (1+\xi)x_2 + (1+\xi)x_3 + (1-\xi)x_4 & -(1-\xi)y_1 - (1+\xi)y_2 + (1+\xi)y_3 + (1-\xi)y_4 \end{bmatrix}$$

Prob.3: Evaluate the shape functions N_1, N_2, N_3 at the interior point P whose coordinates are (3.85,4.8) for the triangular element shown in Fig. (a) below.

Solution: The cartesian co-ordinate at point p can be expressed in terms of nodal coordinates as

$$x = N_1x_1 + N_2x_2 + N_3x_3$$

$$3.85 = 1.5N_1 + 7N_2 + 4N_3 \quad \text{---(i)}$$

$$y = N_1y_1 + N_2y_2 + N_3y_3$$

$$4.8 = 2N_1 + 3.5N_2 + 7N_3 \quad \text{---(ii)}$$

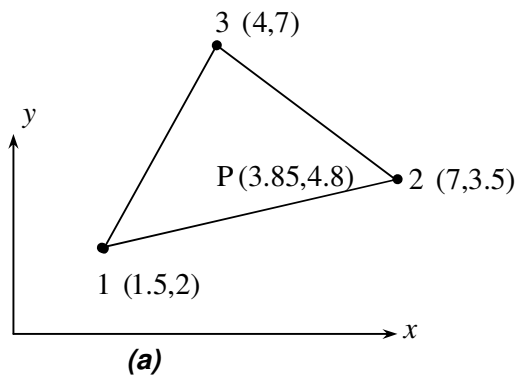
$$\text{also we have } N_1 + N_2 + N_3 = 1 \quad \text{---(iii)}$$

Solving Eqns. (i) – (iii), we get

$$N_1 = 0.3; N_2 = 0.2; N_3 = 0.5$$

Prob.4: determine the Jacobian matrix and B matrix for the triangular element shown in Fig.(a) below.

(VTU, July, 2004)



Solution: The Jacobian matrix is

$$[J] = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}$$

The cartesian co-ordinate at point p can be expressed in terms of nodal coordinates as

$$x = N_1x_1 + N_2x_2 + N_3x_3$$

$$x = 1.5\xi + 7\eta + 4(1-\xi-\eta)$$

$$y = N_1 y_1 + N_2 y_2 + N_3 y_3$$

$$y = 2\xi + 3.5\eta + 7(1 - \xi - \eta)$$

$$\therefore \frac{\partial x}{\partial \xi} = 1.5 - 4 = -2.5 \quad \therefore \frac{\partial y}{\partial \xi} = 2 - 7 = -5.0$$

$$\therefore \frac{\partial x}{\partial \eta} = 7 - 4 = 3.0 \quad \therefore \frac{\partial y}{\partial \eta} = 3.5 - 7 = -3.5$$

$$\text{Hence, } [J] = \begin{bmatrix} -2.5 & -5.0 \\ 3.0 & -3.5 \end{bmatrix}$$

For a triangular element, the strain-displacement matrix is

$$[B]^e = \frac{1}{2A_e} \begin{bmatrix} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \\ c_1 & b_1 & c_2 & b_2 & c_3 & b_3 \end{bmatrix}$$

where, $b_i = y_j - y_k$; $c_i = x_k - x_j$ (suffix $i, j, k = 1, 2, 3$ are in cyclic order).

$$b_1 = y_2 - y_3 = 3.5 - 7 = -3.5$$

$$c_1 = x_3 - x_2 = 4 - 7 = -3$$

$$b_2 = y_3 - y_1 = 7 - 2 = 5$$

$$c_2 = x_1 - x_3 = 1.5 - 4 = -2.5$$

$$b_3 = y_1 - y_2 = 2 - 3.5 = -1.5$$

$$c_3 = x_2 - x_1 = 7 - 1.5 = 5.5$$

$$A_e = \frac{1}{2} \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = \frac{1}{2} (1(x_2 y_3 - x_3 y_2) - x_1(y_3 - y_2) + y_1(x_3 - x_2))$$

Prob.3: The nodal coordinates of a triangular element are shown in Fig. (b). At interior point P the x- coordinate is 3.3 and $N_1=0.3$. Determine N_2 , N_3 and y- coordinate at point P.

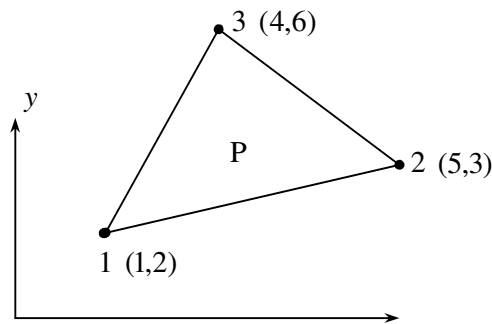


Fig. (b)

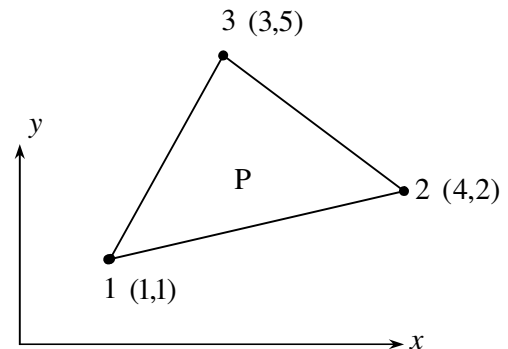


Fig. (c)

Prob.4: For a point P located inside the triangular element shown in Fig.(c), the shape functions N_1 and N_2 are 0.75 and 0.25 respectively. Determine x and y coordinates of point P.

Prob.5: Calculate the displacements at point P whose coordinates are given by (2,3), if the nodal displacements of a triangle are given by

$$u_i = 3\text{mm}; u_j = 4\text{mm}; u_k = 2\text{mm}$$

$$v_i = 2\text{mm}; v_j = 3\text{mm}; v_k = 4\text{mm}$$

Nodes i, j and k of the triangle are given by the coordinates (1,1), (4,1) and (1,5) respectively.

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3 = 3N_1 + 4N_2 + 2N_3 \quad \text{---(a)}$$

$$v = N_1 v_1 + N_2 v_2 + N_3 v_3 = 2N_1 + 3N_2 + 4N_3 \quad \text{---(b)}$$

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3$$

$$2 = 1N_1 + 4N_2 + 1N_3 \quad \text{---(i)}$$

$$y = N_1 y_1 + N_2 y_2 + N_3 y_3$$

$$3 = 1N_1 + 1N_2 + 5N_3 \quad \text{---(ii)}$$

$$N_1 + N_2 + N_3 = 1 \quad \text{---(iii)}$$

Solve Eqns. (i) – (iii) to get N_1, N_2, N_3 and substitute into Eqns. (a) and (b) to obtain u, v

